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Testing for Monotonicity in Expected Asset Returns*

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Abstract

Many postulated relations in finance imply that expected asset returns strictly increase in an underlying characteristic. To examine the validity of such a claim, one needs to take the entire range of the characteristc into account, as is done in the recent proposal of Patton and Timmermann (2010). But their test is only a test for the *direction* of monotonicity, since it requires the relation to be monotonic from the outset: either weakly decreasing under the null or strictly increasing under the alternative. When the relation is non-monotonic or weakly increasing, the test can break down and falsely 'establish' a strictly increasing relation with high probability. We offer some alternative tests that do not share this problem. The behavior of the various tests is illustrated via Monte Carlo studies. We also present empirical applications to real data.

KEYWORDS: Bootstrap; CAPM; Monotonicity tests; Non-monotonic relations.

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1 Introduction

Many postulated relations in finance imply that expected asset returns strictly increase in a certain characteristic. When the assets are equities, examples of such characteristics are CAPM beta, book-to-market, size, momentum, and reversal. When the assets are bonds, examples of such characteristics are maturity and rating quality. The search for new such characteristics is a never-ending quest, partly in hopes of creating novel trading strategies to 'beat the market'.

It is, therefore, of interest to test whether a particular characteristic indeed generates expected asset returns that are strictly increasing. Say there are total of $N + 1$ categories, ordered according to the underlying characteristic. The postulated relation says that as one moves up from one category to the next, then the expected return should strictly increase. (The opposite case of expected asset returns being supposedly monotonically decreasing can be handled analogously by simply multiplying all returns by negative one or, alternatively, by reversing the order of the various return categories considered.)

For a long time, the standard in the field has been to simply test for a difference in expected returns between the highest and the lowest return category. Such a test is easily carried out, since the parameter of interest is univariate, being the difference between two expected values. Therefore, a conventional t-test can be applied, though one has to account for potential serial correlation of returns in computing the standard error that appears in the denominator of the test statistic.

A new test has been proposed recently by Patton and Timmermann (2010), abbreviated by PT henceforth. As they point out, simply testing for a difference in expected returns between the highest and the lowest category can be misleading. It could happen that the relation of expected returns is flat, or even decreasing, for intermediate return categories while a positive difference 'highest minus lowest' still exists. In this case, providing sufficient data are collected, the simple t-test is likely to falsely decide in favor of a strictly monotonic relation. Take the example of five return categories, ordered from lowest to highest, with respective expected returns of 1.0, 1.5, 1.1, 1.4 and 1.6. In this example, the overall relation is non-monotonic, even though the difference 'highest minus lowest' is positive.

Therefore, a more comprehensive approach is needed to establish a strictly monotonic relation over the entire range of return categories. When moving from one category up to the next, the difference in expected returns must be established as significantly positive every time. In other words, all N expected return differentials 'higher minus lower' must be established as significantly greater than zero. Such a test is more complex, since the underlying parameter is now an N-dimensional vector rather than a univariate number.

A natural test statistic for the more comprehensive approach is obtained as follows. Compute individual t-test statistics for each of the N expected return differentials, where in each case the alternative corresponds to the expected return differential being positive. Then take the minimum of the individual test statistics as the overall test statistic. If the resulting min-t statistic is 'sufficiently' large, one decides in favor of a strictly monotonic relation. The statis-

tical question is how to obtain a proper critical value for this test. PT use a bootstrap method, resampling from a certain *worst-case* null distribution. Their proposed method is based on the implicit assumption that if the relation is not strictly increasing, it must be weakly decreasing. That is, if the expected return differentials are not all strictly positive, then they must be all weakly negative (meaning less than or equal to zero). As a result, the test of PT is only a test for the *direction* of monotonicity, not for monotonicity itself, since monotonicity is assumed from the outset.

While the assumption of monotonicity may hold for some applications, it certainly cannot be invoked always. When the true relation is non-monotonic, or even increasing but only weakly instead of strictly, the PT test can break down and no longer successfully control the rejection probability under the null. In fact, as shown in Remark B.1, its size can be arbitrarily close to one: it can falsely decide in favor of the alternative of a strictly monotonic increasing relation with a probability arbitrarily close to one. In this paper, we first discuss this problem of the PT test and then offer some alternative tests that do not share the problem and are, therefore, safer to use in general settings.

Having said this, all the new tests we present have the choice of alternative hypothesis in common with the PT test. The alternative postulates that a monotonic relation (in the sense of being strictly increasing) exists. In other proposals, this postulate becomes the null hypothesis instead; for example, see Wolak (1987, 1989) and also Fama (1984). Such an approach is unnatural, though, if the goal is to establish the existence of a monotonic relation. By not rejecting the null, one can never claim (quantifiable) statistical evidence in favor of the associated hypothesis. Hence, we will not include such tests in our paper. For a theoretical discussion of such tests and some examination of finite-sample performance, see PT.

The remainder of the paper is organized as follows. Section 2 describes the formal setup and the testing problem of interest. Section 3 presents various approaches for designing tests for monotonicity. Section 4 details how the various tests are implemented in practice. Since the available data is assumed to be a multivariate time series, an appropriate bootstrap method is used to calculate critical values in a nonparametric fashion. Section 5 examines finite-sample performance via Monte Carlo studies. Section 6 contains empirical applications to real-life data. Finally, Section 7 concludes. Mathematical results are relegated to the Appendix.

2 Formal setup and testing problem

Our notation generally follows the notation of PT. One observes a strictly stationary time series of return vectors $\boldsymbol{r}_t \equiv (r_{t,0}, r_{t,1}, \ldots, r_{t,N})'$ of dimension $N+1$. The observation period runs from $t = 1$ to $t = T$, so the sample size is given by T. Denote the expected return vector by $\boldsymbol{\mu} \equiv (\mu_0, \mu_1, \ldots, \mu_N)'$ and define the associated expected return differentials as

$$\Delta_i \equiv \mu_i - \mu_{i-1} \quad \text{for } i = 1, \ldots, N \ . \tag{2.1}$$

To avoid any possible confusion, note that the characteristic according to which the $N+1$ return categories are ordered is always assumed to be predetermined and not data-dependent. We also introduce the following notation for the observed return differentials:

$$\boldsymbol{d}_t \equiv (d_{t,1}, \ldots, d_{t,N})' \equiv (r_{t,1} - r_{t,0}, \ldots, r_{t,N+1} - r_{t,N})' \ . \tag{2.2}$$

Therefore, letting $\boldsymbol{\Delta} \equiv (\Delta_1, \ldots \Delta_N)'$, one can also write $\boldsymbol{\Delta} = \mathbb{E}(\boldsymbol{d}_t)$.

The approach proposed by PT specifies a completely flat or weakly decreasing pattern under the null and a strictly increasing pattern under the alternative:

$$H_0 : \Delta_i \leq 0 \text{ for all } i \quad \text{vs.} \quad H_1 : \Delta_i > 0 \text{ for all } i \ . \tag{2.3}$$

Alternatively, these hypotheses can be expressed as

$$H_0 : \Delta_i \leq 0 \text{ for all } i \quad \text{vs.} \quad H_1 : \min_i \Delta_i > 0 \ . \tag{2.4}$$

The problem with this approach is that the alternative is not the negation of the null, at least if the parameter space for $\boldsymbol{\Delta}$ is $\mathbb{R}^N$, the N-dimensional Euclidian space. To see this, one can partition $\mathbb{R}^N$ as follows:

$$\mathbb{R}^N = R_1 \cup R_2 \cup R_3 \ , \tag{2.5}$$

with

$$\begin{aligned} R_1 &\equiv \{\boldsymbol{x} \in \mathbb{R}^N : x_i \leq 0 \text{ for all } i\} \ , \\ R_2 &\equiv \{\boldsymbol{x} \in \mathbb{R}^N : x_i \leq 0 \text{ for some } i \text{ and } x_j > 0 \text{ for some other } j\} \ , \\ \text{and } R_3 &\equiv \{\boldsymbol{x} \in \mathbb{R}^N : x_i > 0 \text{ for all } i\} \ . \end{aligned}$$

R_1 contains all relations that are weakly decreasing, while R_3 contains all relations that are strictly increasing. R_2 contains all other relations. Namely, the interior of R_2,

$$\text{int}(R_2) = \{\boldsymbol{x} \in \mathbb{R}^N : x_i < 0 \text{ for some } i \text{ and } x_j > 0 \text{ for some other } j\} \ , \tag{2.6}$$

contains all relations that are non-monotonic; and the boundary part of R_2,

$$\partial R_2 \cap R_2 = \{\boldsymbol{x} \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \text{ and } x_j = 0 \text{ for some } j \text{ and } x_k > 0 \text{ for some } k\} \ , \tag{2.7}$$

contains all relations that are weakly increasing (but not flat). Note that the origin belongs to the the boundary of R_2,

$$\partial R_2 = \{\boldsymbol{x} \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i\} \ ,$$

but it does not belong to R_2 itself. Indeed, $\mathbf{0} = R_1 \cap \partial R_2$.

The hypotheses of (2.3)–(2.4) can then also be expressed as

$$H_0 : \boldsymbol{\Delta} \in R_1 \quad \text{vs.} \quad H_1 : \boldsymbol{\Delta} \in R_3 \ . \tag{2.8}$$

From the particular expression (2.8), it is now easily appreciated that the alternative is the negation of the null only under the *implicit* assumption

$$\boldsymbol{\Delta} \in R_1 \cup R_3 \ . \tag{2.9}$$

Therefore, it is ruled out a priori that $\boldsymbol{\Delta} \in R_2$, so that the true relation must be necessarily monotonic; the only feature in question is the *direction* of the monotonicity.

Remark 2.1 In the light of the previous discussion, the following statement in the conclusions of PT needs some clarification:

> In this paper we propose a test that reveals whether a null hypothesis of no systematic relation can be rejected in favor of a monotonic relation predicted by economic theory.

Since a systematic relation, in this context, is the same thing as a monotonic relation, a better formulation would have been as follows: "In this paper, we propose a test that reveals whether a null hypothesis of a weakly decreasing relation can be rejected in favor of a strictly increasing relation predicted by economic theory." In fact, this latter formulation is consistent with previous statements in PT such as "the approach [. . .] specifies a flat or weakly decreasing pattern under the null hypothesis and strictly increasing pattern under the alternative"; see their Section 2.3. ■

If the goal is to establish strict monotonicity of expected asset returns and one allows for $\boldsymbol{\Delta} \in \mathbb{R}^N$ a priori, then the hypotheses must be formulated as follows instead:

$$H_0 : \boldsymbol{\Delta} \in R_1 \cup R_2 \quad \text{vs.} \quad H_1 : \boldsymbol{\Delta} \in R_3 \tag{2.10}$$

and can also be expressed as

$$H_0 : \min_i \Delta_i \leq 0 \quad \text{vs.} \quad H_1 : \min_i \Delta_i > 0 \ . \tag{2.11}$$

The distinction between (2.8) and (2.10) is of crucial importance for the proper implementation of the test. This will be explained in detail in the next section.

Remark 2.2 Our goal is to find tests that establish strict monotonicity. Although it may be economically of interest to be able to establish the weaker conclusion of weak or strict monotonicity, it is not clear how to set up such testing problem statistically. ■

Remark 2.3 Postulated relations that imply strictly increasing expected returns can be based either on economic theory or on empirical findings (that is, so-called market anomalies). In the context of equity returns, an example of the former is the CAPM beta while examples of the latter are book-to-market, size, momentum, and reversal. (There may exist certain models that also imply monotonicity of expected returns for some of the latter examples. But such

models have often been derived only *after* observing market anomalies in data. As a result, in the finance community, they do not enjoy the standing of the CAPM.)

The CAPM postulates that the expected (excess) return of a stock is a linear function of its CAPM beta, with the slope of the linear function being positive. Assume now that stocks are grouped into deciles, say, based on their CAPM beta. Then, if the linearity of the postulated relation is taken for granted, the expected returns of the ten categories will be either strictly increasing (namely if the slope of the linear relation is indeed positive) or weakly decreasing (namely if the slope of the linear relation is non-positive instead). In this case, the correct formulation of the testing problem is given by (2.8). On the other hand, if the linearity of the postulated relation is not taken for granted, and it also cannot be ruled out that the relation is non-monotonic or weakly increasing, the correct formulation of the testing problem is given by (2.10) instead.

The same reasoning applies to any other model that postulates that the expected (excess) return of a stock is a linear function of a certain characteristic. Though if the linearity of the postulated relation is taken for granted, one would suspect that an even more powerful test could be devised compared to the test of PT. To make an analogy, consider studying the effect of years of schooling on wages in a regression model. If the form of the relation is not clear, a standard approach is to group years of schooling into a finite number of categories (0–5, 6–10, 11–15, etc.) and to include a dummy variable per category into the regression model. One then would have to show that the corresponding coefficients are strictly increasing to establish a strictly monotonic effect. On the other hand, if the relation is taken for granted as linear, a simpler (and more powerful) approach is to include years of schooling as a numerical regressor into the linear regression model and to then establish the corresponding slope as positive.

When the postulated relation is based on an empirical market anomaly rather than on economic theory, it seems prudent to also allow for a non-monotonic or weakly increasing relation under the null. In this case, the correct formulation of the testing problem is given by (2.10). ■

3 Designing tests for monotonicity in expected asset returns

It is assumed that estimators $\hat{\Delta}_{T,i}$ for the individual expected return differentials Δ_i are available. Here, the subscript T makes it explicit that the estimators depend on the given sample size T. The natural estimator for Δ_i is simply the corresponding sample mean

$$\hat{\Delta}_{T,i} \equiv \bar{d}_{T,i} \equiv \frac{1}{T}\sum_{i=1}^{T} d_{t,i} \ . \tag{3.1}$$

But the theory also allows for other choices $\hat{\Delta}_{T,i}$, such as robust estimators of means; for example, see Scherer and Martin (2005, Section 6.3).

Both expressions (2.4) and (2.11) initially suggest the following (overall) test statistic

$$\hat{\Delta}_T^{min} \equiv \min_i \hat{\Delta}_{T,i} \ . \tag{3.2}$$

But, as do PT, we instead advocate the use of studentized individual test statistics. The main reason for studentization is to make sure that the individual test statistics are on the same scale, which can lead to large power gains; for example, see Hansen (2005) and Romano and Wolf (2005). The individual test statistic corresponding to Δ_i is then given by

$$t_{T,i} \equiv \frac{\hat{\Delta}_{T,i}}{\hat{\sigma}_{T,i}} \ , \tag{3.3}$$

where $\hat{\sigma}_{T,i}$ is a standard error for $\hat{\Delta}_{T,i}$. If the data constitute a time series, the standard error must be robust in that it must account for serial correlation; for example, one can use a HAC standard error as described in Andrews (1991) or a prewhitened HAC standard error as described in Andrews and Monahan (1992).

The following shall be assumed throughout, for $i = 1, \ldots N$: the estimator $\hat{\Delta}_{T,i}$ is shift-equivariant and its standard error $\hat{\sigma}_{T,i}$ is shift-invariant. That is, if the same constant c is added to all data points $d_{1,i}, \ldots, d_{T,i}$, the estimator $\hat{\Delta}_{T,i}$ is shifted by the same amount c, while its standard error $\hat{\sigma}_{T,i}$ remains unchanged. This assumption holds true for any reasonable choice of estimator and standard error.

The (overall) test statistic is then defined as

$$t_T^{min} \equiv \min_i t_{T,i} \ . \tag{3.4}$$

Regardless of the choice of test statistic, $\hat{\Delta}_T^{min}$ or t_T^{min}, one decides in favor of the alternative if the observed value of the test statistic is 'sufficiently' large. The difficulty lies in determining what constitutes 'sufficiently' large. That is, how does one determine a proper critical value for a level α test? In the remainder of the paper, we shall restrict attention to the test statistic t_T^{min}; the issues are analogous for the alternative choice of test statistic $\hat{\Delta}_T^{min}$.

Remark 3.1 All the tests for testing problem (2.10) that will be discussed have the following feature: a necessary condition for deciding in favor of H_1 is that $t_T^{min} > 0$, which happens if and only if $\min_i \hat{\Delta}_{T,i} > 0$. This is a reasonable restriction accepted by the majority of the statistics community: to reject H_0 in favor of the hypothesis H_1 that the true parameter lies in a certain region, it is necessary that the observed parameter estimate lie in that region to begin with. For our purposes, to establish that $\boldsymbol{\Delta} \in R_3$ with any statistical significance, it is necessary that $\hat{\boldsymbol{\Delta}}_T \in R_3$ to begin with.

An additional motivation for this feature is the concept of a monotone test; the term *monotone* here has a different meaning compared to the relation of expected returns being *monotonic*. In the context of testing problem (2.10), assume a test rejects H_0 based on a vector of individual test statistics $\boldsymbol{t}_T \equiv (t_{T,1}, \ldots, t_{T,N})'$. Then the test is said to be *monotone* if it also rejects H_0 based on any other vector of individual test statistics $\tilde{\boldsymbol{t}}_T \equiv (\tilde{t}_{T,1}, \ldots, \tilde{t}_{T,N})'$ that satisfies $\tilde{\boldsymbol{t}}_T \geq \boldsymbol{t}_T$. Such a requirement appears reasonable: if $\tilde{\boldsymbol{t}}_T \geq \boldsymbol{t}_T$, then $\tilde{\boldsymbol{t}}_T$ should be considered at least as significant against H_0 as $\boldsymbol{t}_T$. If the requirement of a monotone test is adopted, it can be shown that the feature of $t_T^{min} > 0$ being a necessary condition for rejection of H_0 is rather innocuous; see the end of Subsection 3.1.

On the other hand, the MR test of PT for testing problem (2.8) can decide in favor of $\boldsymbol{\Delta} \in R_3$ when $\hat{\boldsymbol{\Delta}}_T \in R_2$; for example, see Subsection 6.1. This feature is due to the artifact that $\boldsymbol{\Delta} \in R_2$ is ruled out a priori by the assumptions of the MR test. ■

The general strategy to compute the critical value for a test against a one-sided (larger than) alternative at significance level α is to derive the sampling distribution of the test statistic 'under the null' and to then take the $1-\alpha$ quantile of this derived null sampling distribution as the critical value.

Whether the testing problem is (2.8) or (2.10), the null hypothesis is *composite*, meaning that there are (infinitely) many parameters $\boldsymbol{\Delta}$ in the null space. Therefore, it is not immediately clear under which particular value of $\boldsymbol{\Delta}$ one should derive the sampling distribution 'under the null'. One can deal with this dilemma in three ways: use the worst-case parameter under the null; try and guess the null parameter; or bound the true null parameter.

In the remainder of our discussion, we consider the data generating process (DGP) to be unknown but fixed, with the possible exception of $\boldsymbol{\Delta}$. For example, when we talk about a worst-case distribution, as discussed below, we mean worst case with respect to $\boldsymbol{\Delta}$ only but not with respect to other features of the DGP, such as the correlation structure within the vector of return differentials $\boldsymbol{d}_t$, the various marginal distributions of the individual return differentials $d_{t,i}$, or the dependence structure of the $\boldsymbol{d}_t$ over time. All such other features are considered (unknown but) fixed.

More specifically, one can regard the unknown joint distribution of $\hat{\boldsymbol{\Delta}}_T$ as that of $\boldsymbol{X} + \boldsymbol{\Delta}$, where $\boldsymbol{X}$ is a random vector with a fixed distribution and $\boldsymbol{\Delta}$ is the unknown parameter. For example, $\boldsymbol{X}$ could have the multivariate normal distribution with mean vector $\boldsymbol{0}$ and (unknown but) fixed covariance matrix $\boldsymbol{\Sigma}$; or it can be multivariate-t; or something else. Regardless, under our maintained assumption that $\hat{\Delta}_{T,i}$ is shift-equivariant while $\hat{\sigma}_{T,i}$ is shift-invariant for all $i = 1, \ldots, N$, it then follows that larger values of $\boldsymbol{\Delta}$ lead to larger values of the test statistic t_T^{min}. In particular, if $\boldsymbol{\Delta}_1 \leq \boldsymbol{\Delta}_2$ (with vector inequalities interpreted componentwise), then $\mathbb{P}_{\boldsymbol{\Delta}_1}\{t_T^{min} > c\} \leq \mathbb{P}_{\boldsymbol{\Delta}_2}\{t_T^{min} > c\}$, for any real number c.

3.1 Worst-case approaches

The worst case under the null corresponds to the parameter $\boldsymbol{\Delta}_0$ in the null space that results in the largest critical value possible. Specifically, if we consider the $1-\alpha$ quantile of the distribution of a given test statistic under $\boldsymbol{\Delta}_0$, then the most conservative approach is to take the largest such critical value as $\boldsymbol{\Delta}_0$ ranges over the null hypothesis parameter space. The corresponding *worst-case* parameter $\boldsymbol{\Delta}_0$ lies on the boundary between the null and the alternative space. (In statistical lingo, it gives rise to the *least favorable distribution*, where the term 'favorable' is to be understood with respect to the ability to reject false hypotheses: the larger the critical value of the test, the smaller is its power against any given alternative.)

The motivation in doing so is simple: if one uses the worst-case parameter under the null to derive the critical value, then, by definition, the test will work for any parameter under the

null in the sense that rejection probability will be at most α. This is because, by definition, the critical value under any other parameter in the null space is at most as large as the critical value derived under the worst-case parameter.

Crucially, the worst-case parameter under the null depends on the choice of testing problem: (2.8) or (2.10). For testing problem (2.8), it is easy to see that the worst-case parameter is given by

$$\boldsymbol{\Delta}_0^{\mathrm{WC},(2.8)} \equiv \mathbf{0} \equiv (0, 0, \ldots, 0)' \,. \tag{3.5}$$

The resulting test is called the MR test (for monotonic-relation test) by PT.

But this is no longer true for testing problem (2.10). Consider the special case $N = 2$. For any positive number δ, both the parameters $(\delta, 0)$ and $(0, \delta)$ will lead to a larger critical value compared to the parameter $(0, 0)$. (However, which of the two leads to the largest critical value is not necessarily clear.) Furthermore, increasing δ will increase the two critical values. Taking this logic to the limit, one realizes that the worst-case parameter is one of the following two: $(\infty, 0)$ or $(0, \infty)$.

Remark 3.2 Any parameter $\boldsymbol{\Delta}$ can only have finite entries. What we mean by the worst-case parameter being equal to $(\infty, 0)$, say, is the following. The critical value under the parameter $(\infty, 0)$ is obtained as the limit of the critical value under $(\delta, 0)$, as δ tends to infinity. For all practical purposes, the critical value can be obtained under $(\delta^{\mathrm{Big}}, 0)$ where δ^{Big} is a very large number, such as $\delta^{\mathrm{Big}} = 10^6$. ■

For a general dimension N, the worst-case parameter for testing problem (2.10) is of the form

$$\boldsymbol{\Delta}_0^{\mathrm{WC},(2.10)} \equiv (\infty, \ldots, \infty, 0, \infty, \ldots, \infty)' \,. \tag{3.6}$$

That is, one of the entries is zero and all the other entries are infinity. In case it is not clear a priori which particular position of the zero entry leads to the largest possible critical value, one has to try out all possible positions in principle. However, as will be seen, under mild regularity conditions, all such worst-case parameters lead to an asymptotic critical value of $z_{1-\alpha}$, the $1-\alpha$ quantile of the standard normal distribution.

We call the resulting test the Cons test, short for conservative test. A connection to the generalized likelihood ratio test for testing problem (2.10) is given in Appendix A.

Recall the concept of a monotone test from Remark 3.1. It can be shown that for testing problem (2.10), in an approximate setup, the Cons test is uniformly most powerful (UMP) among all monotone level α tests; see Appendix B. As the critical value of the Cons test is positive, this provides another justification to restrict attention to tests that require $t_T^{min} > 0$ as a necessary condition to reject H_0 in testing problem (2.10).

3.2 Constrained estimation of the null parameter

By design, using the worst-case parameter to derive the critical value of a test leads to conservative inference in general. That is, the rejection probability under an arbitrary parameter in

the null hypothesis parameter space will often be less than the significance level of the test.

Take the example of testing problem (2.10) with $N = 2$. If the worst-case parameter is $(\infty, 0)$ but the true parameter is $(1, 0)$, say, than the rejection probability is smaller than α.

Therefore, it would be better to compute the critical value under the true parameter $(1, 0)$. The dilemma is that the true parameter is not known. It is then tempting to estimate the true parameter from the data and to use a corresponding parameter to compute the critical value rather than to use the worst-case parameter. In doing so, the estimate must be constrained to lie in the null parameter space.

One observes $\hat{\mathbf{\Delta}}_T \equiv (\hat{\Delta}_{T,1}, \ldots, \hat{\Delta}_{T,N})'$. As discussed before in Remark 3.1, we only have to consider the case of $\hat{\Delta}_T^{min} > 0$, since any 'reasonable' test for testing problem (2.10) will not reject if $\hat{\Delta}_T^{min} \leq 0$. A constrained estimate for the null parameter is then of the form

$$\mathbf{\Delta}_0^{\text{Est}} \equiv (\hat{\Delta}_{T,1}, \ldots, \hat{\Delta}_{T,i-1}, 0, \hat{\Delta}_{T,i+1}, \ldots, \hat{\Delta}_{T,N})' \; . \tag{3.7}$$

It is not clear a priori which position for the zero entry will lead to the largest critical value, so in principle one has to try out all N possibilities. Since this approach is based on the observed estimate $\hat{\mathbf{\Delta}}_T$ rather than the true parameter $\mathbf{\Delta}$, one cannot guarantee that the resulting test will work (in the sense of ensuring rejection probability of at most α for any null parameter) in finite samples.

We will call the resulting test the Constrained-Estimate (CE) test.

3.3 Constrained bounding of the null parameter

A more conservative approach first constructs an upper joint confidence region for $\mathbf{\Delta}$ and then extracts the null parameter from the corresponding upper limits rather than from the point estimate $\hat{\mathbf{\Delta}}_T$. Crucially, to ensure that the resulting test works well in finite samples, one has to downward adjust the significance level of the test depending on the confidence level used for the joint confidence region.

More specifically, consider a joint confidence region for $\mathbf{\Delta}$ at confidence level $1 - \beta$ of the form

$$(-\infty, \hat{u}_{T,1}] \times (-\infty, \hat{u}_{T,2}] \times \ldots \times (-\infty, \hat{u}_{T,N}] \; . \tag{3.8}$$

For example, the vector of upper limits $\hat{\boldsymbol{u}}_T \equiv (\hat{u}_{T,1}, \hat{u}_{T,2}, \ldots, \hat{u}_{T,N})'$ can be computed by the single-step method described in Romano and Wolf (2005); for completeness, this method is briefly summarized in Appendix C. It satisfies $\hat{u}_{T,i} > \hat{\Delta}_{T,i}$, for all i. The constrained bounded parameter then lies on the intersection of the region (3.8) together with the null hypothesis parameter space $R_1 \cup R_2$ and, therefore, is of the form

$$\mathbf{\Delta}_0^{\text{Bound}} \equiv (\hat{u}_{T,1}, \ldots, \hat{u}_{T,i-1}, 0, \hat{u}_{T,i+1}, \ldots, \hat{u}_{T,N})' \; . \tag{3.9}$$

Again, it is not clear a priori which position for the zero entry will lead to the largest critical value, so in principle one has to try out all N possibilities. Crucially, the critical value is now computed as the $1 - \alpha + \beta$ quantile of the corresponding null sampling distribution of t_T^{min}

to ensure the overall validity of the test in finite samples. The intuition here is that with probability of (at most) β the confidence region (3.8) will not contain the true parameter $\boldsymbol{\Delta}$; and one must then adjust for this fact by decreasing the nominal significance level of the test from α to $\alpha - \beta$. In particular, one must choose $\beta < \alpha$ in the computation of the confidence region (3.8).

We will call the resulting test the Two-Step test, due to its two-step nature. Related two-step inference procedures have been previously suggested by Loh (1985), Romano and Wolf (2000), Hansen (2003), Moon and Schorfheide (2009), and McCloskey (2012).

4 Implementing tests for monotonicity in expected asset returns

Having chosen a specific null parameter $\boldsymbol{\Delta}_0$, by any one of the forms (3.5), (3.6), (3.7), or (3.9), one is left to derive or approximate the implied sampling distribution of the test statistic t_T^{min}, and to then take the appropriate quantile as the critical value of the test.

It is not possible to derive the implied sampling distribution analytically. As a feasible solution, the distribution can be approximated via the bootstrap. To this end, create a bootstrap data set $\{\boldsymbol{r}_1^*, \boldsymbol{r}_2^*, \ldots, \boldsymbol{r}_T^*\}$ by resampling from the observed data $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_T\}$. Denote the estimator of Δ_i computed from the bootstrap data by $\hat{\Delta}_{T,i}^*$ and let $\boldsymbol{\Delta}_T^* \equiv (\hat{\Delta}_{T,1}^*, \ldots, \hat{\Delta}_{T,N}^*)'$. Similarly, denote the standard error for $\hat{\Delta}_{T,i}^*$ computed from the bootstrap data by $\hat{\sigma}_{T,i}^*$. The following algorithm details how the critical value of the test is computed.

Algorithm 4.1 (Computation of the critical value via the bootstrap)

- Generate bootstrap data $\{\boldsymbol{r}_1^*, \boldsymbol{r}_2^*, \ldots, \boldsymbol{r}_T^*\}$ and compute statistics $\hat{\Delta}_{T,i}^*$ and $\hat{\sigma}_{T,i}^*$ from these data, for $i = 1, \ldots, N$.
- Compute $t_{T,i}^*$, for $i = 1, \ldots, N$, defined as:

$$t_{T,i}^* \equiv \frac{\hat{\Delta}_{T,i}^* - \hat{\Delta}_{T,i} + \Delta_{0,i}}{\hat{\sigma}_{T,i}^*} \,. \tag{4.1}$$

- Compute $t_T^{min,*} \equiv \min_i t_{T,i}^*$.
- Repeat this process B times, resulting in statistics $t_{T,1}^{min,*}, \ldots, t_{T,B}^{min,*}$.
- The empirical distribution of these statistics $t_{T,1}^{min,*}, \ldots, t_{T,B}^{min,*}$ is the bootstrap approximation to the sampling distribution of t_T^{min} under the parameter $\boldsymbol{\Delta}_0$.
- In particular, the corresponding $1 - \alpha$ empirical quantile serves as the critical value of the test; with the exception of the Two-Step test, where one takes the corresponding $1 - \alpha + \beta$ empirical quantile instead.

Remark 4.1 Which particular bootstrap method should be used to generate the bootstrap data $\{\boldsymbol{r}_1^*, \boldsymbol{r}_2^*, \ldots, \boldsymbol{r}_T^*\}$ from the observed data $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_T\}$ depends on the underlying DGP. If the original data are i.i.d., one can use the standard bootstrap of Efron (1979). If the original data constitute a time series, one needs to employ a proper time series bootstrap and

several choices are available; for example, see Lahiri (2003). Romano and Wolf (2005) suggest the circular block bootstrap of Politis and Romano (1992) while PT suggest the stationary bootstrap of Politis and Romano (1994). Either way, one must preserve the cross-sectional dependence within the return vectors $\boldsymbol{r}_t$, so these 'units' are never broken up: for example, if the original data are i.i.d., the standard bootstrap of Efron (1979) resamples the $\boldsymbol{r}_t^*$ in an i.i.d. fashion from the empirical distribution of the observed return vectors $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T$.

It is important to point out that we simply 'resample from the observed data'. This is because we add the number $(\Delta_{0,i} - \hat{\Delta}_{T,i})$ to the bootstrap estimate $\hat{\Delta}^*_{T,i}$ in the numerator of the bootstrap test statistic (4.1) to approximate the sampling distribution of t_T^{min} under $\boldsymbol{\Delta}_0$ in the end, as desired. This approach is (asymptotically) equivalent to instead 'resampling from some null-enforced data', where $\boldsymbol{\Delta}_0$ is true parameter, and to then computing the bootstrap test statistic as

$$t^*_{T,i} \equiv \frac{\hat{\Delta}^*_{T,i}}{\hat{\sigma}^*_{T,i}} \ . \tag{4.2}$$

The latter approach is more complicated and could be carried out by some empirical likelihood method; for example, see Owen (2001). ■

Remark 4.2 There exists an alternative way to implement the Cons test. Assume the position of the zero entry in the worst-case parameter (3.6) is given by i; all the other entries are equal to infinity. In every bootstrap repetition, the smallest bootstrap t-statistic will then correspond to entry i, since all other bootstrap t-statistics will be equal to infinity (with probability one):

$$t_T^{min,*} = t^*_{T,i} \ . \tag{4.3}$$

Therefore, the critical value can simply be obtained by using a univariate bootstrap on the ith return differentials $\{d_{1,i}, \ldots, d_{T,i}\}$ for the testing problem

$$H_0 : \Delta_i \leq 0 \quad \text{vs.} \quad \Delta_i > 0 \ . \tag{4.4}$$

This must be done for each $i = 1, \ldots, N$ (or just use the resampling statistics obtained by Algorithm 4.1 marginally). In the end, the largest of the resulting N critical values needs to be employed.

On the other hand, if the sample size is very large and finite-sample considerations are not really of concern, one can simply take as the critical value the constant $z_{1-\alpha}$, the $(1-\alpha)$ quantile of $N(0,1)$, completely avoiding the application of a bootstrap; see Remark 4.4 ■

So far Algorithm 4.1 is merely a 'recipe'. Certain conditions are needed to ensure that the resulting critical value obtained by the bootstrap is a valid approximation. To this end, we will make use of the following high-level assumptions.

(A1) $\sqrt{T}(\hat{\boldsymbol{\Delta}} - \boldsymbol{\Delta})$ converges in distribution to $N(\boldsymbol{0}, \boldsymbol{\Omega})$, for some positive definite $N \times N$ matrix $\boldsymbol{\Omega}$ with typical element $\omega_{i,j}$.

(A2) The bootstrap consistently estimates this limiting distribution, namely $\sqrt{T}(\hat{\boldsymbol{\Delta}}^* - \hat{\boldsymbol{\Delta}})$ converges in distribution to $N(\mathbf{0}, \boldsymbol{\Omega})$ in probability.

(A3) Both $\sqrt{T}\hat{\sigma}_{T,i}$ and $\sqrt{T}\hat{\sigma}^*_{T,i}$ converge in probability to $\sqrt{\omega_{i,i}}$, for $i = 1, \ldots, N$.

These high-level assumptions are rather weak and have been verified for many applications of interest in the statistics literature.

One can then show the following theorem, the proof of which can be found in Appendix D:

Theorem 4.1 *Assume (A1)–(A3) and compute the critical value of the tests as detailed in Algorithm 4.1. Then, as $T \to \infty$:*

(ia) For any $\boldsymbol{\Delta} \in R_1$ and for any $\boldsymbol{\Delta} \in int(R_2)$, as defined in (2.6), $\min_i \Delta_i < 0$, the limiting rejection probability of the MR test is no larger than α.

(ib) For any $\boldsymbol{\Delta}$ on the boundary part of R_2, as defined in (2.7), the limiting rejection probability of the MR test is strictly larger than α.

(ic) For any $\boldsymbol{\Delta} \in R_3$, the limiting rejection probability of the MR test is one.

(iia) For any $\boldsymbol{\Delta} \in R_1 \cup R_2$, the limiting rejection probabilities of the Cons, CE, and Two-Step tests are no larger than α.

(iib) For any $\boldsymbol{\Delta} \in R_3$, the limiting rejection probabilities of the Cons, CE, and Two-Step tests are one.

Remark 4.3 An important implication for applied work stems from part (ib) of the theorem. If $\boldsymbol{\Delta}$ lies on the boundary part of R_2, then the limiting rejection probability of the test is strictly larger than α; such a $\boldsymbol{\Delta}$ corresponds to a relation that is only weakly increasing (but not flat). In the special case where the test statistics are asymptotically uncorrelated, the probability of a type 1 error can, approximately, be as large as $\alpha^{1/N}$; see Remark B.1 in Appendix B. So even when $\boldsymbol{\Delta} \in R_2$, the test can decide in favor of $\boldsymbol{\Delta} \in R_3$ with high probability. ■

Remark 4.4 The worst-case parameter $\boldsymbol{\Delta}_0^{\mathrm{WC},(2.10)}$ for the Cons test is of the form (3.6), that is, all entries are infinity apart from a single zero entry. But then, under Assumptions (A1)–(A3), the limiting critical value of the test is simply $z_{1-\alpha}$, the $1-\alpha$ quantile of the standard normal distribution, regardless of the position of the zero entry; see the proof of Theorem 4.1(ib).

Since the bootstrap critical value is valid asymptotically only in the first place, one can simply use $z_{1-\alpha}$ as the critical value of the test, foregoing any application of the bootstrap. Indeed, the two approaches are equivalent to first order because the bootstrap critical value converges to $z_{1-\alpha}$ in probability. However, it should be pointed out that the computation of the critical value via the bootstrap typically results in better finite-sample properties of the test, as the bootstrap is better able to capture skewness; for example, see Hall (1992). ■

Remark 4.5 At first, one might conclude from Theorem 4.1 that the failure of the MR test occurs only when $\boldsymbol{\Delta}$ lies on the boundary part of R_2. Indeed, if $\boldsymbol{\Delta}$ lies in the interior of R_2,

then any component of $\boldsymbol{\Delta}$ with $\Delta_i < 0$ will have a corresponding t-statistic $t_{T,i}$ tending in probability to $-\infty$. The reason is that, with $\boldsymbol{\Delta}$ considered fixed and $T \to \infty$, any point on the boundary can be perfectly distinguished (just as any alternative has power tending to one). However, if one considers local asymptotics, then the limiting rejection probability under a sequence of parameters $\boldsymbol{\Delta}$ in the interior of R_2, but getting closer to the boundary, can easily exceed the nominal level α. To put it another way, the supremum of the rejection probability over $\boldsymbol{\Delta} \in \text{int}(R_2)$ will exceed α. This phenomenon occurs because Theorem 4.1 only considers pointwise asymptotics (with the data generating distribution fixed as $T \to \infty$), but to gain further understanding it is useful to consider uniform asymptotics as well.

The importance of uniform asymptotics is stressed in Romano and Shaikh (2013), and the many references therein. Although certain technical aspects of uniform asymptotics are beyond the scope of this paper, we provide some argument in this remark. The basic idea is quite clear: if the rejection probability on the boundary part is much larger than α for a given sample size T, then by continuity, it must be bigger than α 'near' the boundary part as well!

More specifically, consider a sequence of parameters $\{\boldsymbol{\Delta}_T\}$ of the form

$$\boldsymbol{\Delta}_T \equiv \boldsymbol{\Delta}_0 + \frac{\boldsymbol{\gamma}}{\sqrt{T}} \,, \tag{4.5}$$

where $\boldsymbol{\Delta}_0$ is a fixed vector on the boundary part of R_2 and $\boldsymbol{\gamma}$ is a fixed vector with at least one component $\gamma_i < 0$ among the components $\Delta_{0,i}$ of $\boldsymbol{\Delta}_0$ that are zero. Thus, for all T, $\boldsymbol{\Delta}_T \in \text{int}(R_2)$. Although such a sequence is getting closer and closer to the boundary, the important point is that such *local* (or *contiguous*) alternatives are such that it is impossible to devise a statistical test that can distinguish perfectly between such points and points on the boundary, even as $T \to \infty$. Such a local approach is both natural and necessary for the asymptotics to have a more reliable bearing on the finite-sample testing problem. Now, if assumption (A1) holds, it then also typically holds (using contiguity arguments) that $\sqrt{T}(\hat{\boldsymbol{\Delta}}_T - \boldsymbol{\Delta}_T)$ converges in distribution to $N(\mathbf{0}, \boldsymbol{\Omega})$, so that the convergence is locally uniform with respect to the parameter $\boldsymbol{\Delta}$. If this is the case, then arguing as in the proof of Theorem 4.1,

$$t_{T,i} = \nu_{T,i} + \frac{\Delta_{T,i}}{\hat{\sigma}_{T,i}} = \nu_{T,i} + \frac{\Delta_{0,i} + \gamma_i/\sqrt{T}}{\hat{\sigma}_{T,i}} \,.$$

Hence, if $\Delta_{0,i} > 0$, then $t_{T,i} \to \infty$ in probability, while if $\Delta_{0,i} = 0$, then $\gamma_i/(\sqrt{T}\hat{\sigma}_{T,i})$ tends in probability to $\gamma_i/\pi_{i,i}$ and $t_{T,i}$ converges in distribution to $N(\gamma_i/\pi_{i,i}, \pi_{i,i})$; here $\pi_{i,i}$ is the ith diagonal element of the correlation matrix $\boldsymbol{\Pi}$ defined as in (D.1). As in the proof of Theorem 4.1, let $I(\boldsymbol{\Delta}_0) = \{i : \Delta_{0,i} = 0\}$. Thus, the limiting rejection probability under such a sequence $\{\boldsymbol{\Delta}_T\}$ can be expressed as

$$P\left\{ \min_{i \in I(\boldsymbol{\Delta}_0)} \left[y_i + \frac{\gamma_i}{\pi_{i,i}} \right] > c_{1-\alpha} \right\} \,, \tag{4.6}$$

where $y \sim N(\mathbf{0}, \boldsymbol{\Pi})$, and $c_{1-\alpha}$ is the $1-\alpha$ quantile of the distribution of $\min_i y_i$. For $\min_{i \in I(\boldsymbol{\Delta}_0)} \gamma_i$ close to zero, the limiting rejection probability under such a sequence $\{\boldsymbol{\Delta}_T\}$

(by continuity) is close to the limiting rejection probability for a parameter on the boundary part of R_2, which by Theorem 4.1 exceeds α as long as $\max_{i \in I(\boldsymbol{\Delta}_0)} \gamma_i > 0$. For example, let $\boldsymbol{\Delta}_0 = (\Delta_{0,1}, \ldots, \Delta_{0,k}, 0, \ldots, 0)'$ with $1 \leq k \leq N-1$ and $\min_{1 \leq i \leq k} \Delta_{0,i} > 0$, so $\boldsymbol{\Delta}_0$ is on the boundary part of R_2. If we choose $\boldsymbol{\gamma} = (0, \ldots, 0, \gamma_N)'$ with $\gamma_N < 0$, then for a sequence $\{\Delta_T\}$ of the form (4.5), Δ_T will lie in the interior of R_2 for all T. If, furthermore, we choose $|\gamma_N|$ small, then the probability in (4.6) will exceed α.

It is difficult to determine how large the probability in equation (4.6) can actually get, as one would have to maximize jointly with respect to $\boldsymbol{\Delta}_0$, $\boldsymbol{\gamma}$, and $\boldsymbol{\Pi}$. By taking $k = N-1$, $\gamma_N \to 0$, and $\boldsymbol{\Pi} = \boldsymbol{I}_N$ (that is, the N-dimensional identity matrix) in the previous example, a lower bound is given by $\alpha^{1/N}$; see Remark B.1. When N is large, this lower bound is close to one.

Consequently, one should expect that the MR test can be anti-conservative in finite samples even for some null parameters $\boldsymbol{\Delta}$ in the interior of R_2; such is indeed the case, as seen in Section 5.

On the other hand, under an additional mild uniform integrability condition on the return differentials $\boldsymbol{d}_t$, it can be shown that the more conservative approaches of the Cons test and the Two-Step test do not suffer this lack of uniformity and asymptotically control the size of the test uniformly over the whole null hypothesis parameter space. Indeed, for the Cons test, even under sequences of null parameters $\{\boldsymbol{\Delta}_T\}$ of the form (4.5), the critical value still converges in probability to $z_{1-\alpha}$. A formal argument can be based on the recent work of Romano and Shaikh (2013). ■

5 Monte Carlo study

This section examines the finite-sample properties of the various tests via Monte Carlo studies; both in terms of controlling the probability of a type 1 error under the null and in terms of power.

The following tests are included in the study:

- **(MR)** The MR test of PT described in Subsection 3.1.
- **(CE)** The CE test described in Subsection 3.2.
- **(TwoStep)** The Two-Step test described in Subsection 3.3.
- **(Cons)** The Cons test described in Subsection 3.1; its critical value is computed using the bootstrap rather than simply taking $z_{1-\alpha}$; see Remark 4.2.

We use $\alpha = 0.05$ as the significance level, and $\beta = 0.01$ for the Two-Step test. The sample size is $T = 120$ always. All empirical rejection probabilities are based on 20,000 simulations. The number of bootstrap repetitions is $B = 499$. Return differential vectors $\boldsymbol{d}_t$ are generated in i.i.d. fashion and, accordingly, the standard bootstrap of Efron (1979) is employed.[1]

[1]Since the purpose of this section is to study the *relative* performance of the various tests, there is no need to also consider time series data with all the resulting complex issues of which time series bootstrap to employ.

The individual test statistics $t_{T,i}$ are computed as follows

$$t_{T,i} \equiv \frac{\hat{\Delta}_{T,i}}{\hat{\sigma}_{T,i}} \quad \text{where} \quad \hat{\Delta}_{T,i} \equiv \bar{d}_{T,i} \equiv \frac{1}{T}\sum_{t=1}^{T} d_{t,i} \quad \text{and} \quad \hat{\sigma}^2_{T,i} \equiv \frac{1}{T-1}\sum_{t=1}^{T}(d_{t,i} - \bar{d}_{T,i})^2 \ . \tag{5.1}$$

5.1 Null behavior for $N = 2$

We start with the extreme case of $N = 2$ to more easily judge the effect of correlation within a return differential vector. The return differentials are generated as

$$\boldsymbol{d}_t = \begin{pmatrix} d_{t,1} \\ d_{t,2} \end{pmatrix} \sim N\left(\begin{pmatrix} \Delta_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \ , \tag{5.2}$$

with $\Delta_1 \in \{0, 0.025, 0.05, \ldots, 0.45, 0.475, 0.5\}$ and $\rho \in \{-0.5, 0.0, 0.5\}$.

The parameter $\boldsymbol{\Delta}$ lies on the boundary of R_2 always. For the special case of $\Delta_1 = 0$, it also lies on the boundary of R_1, too. But it never lies in R_3; that is, the alternative of strict monotonicity is never true.

The empirical rejections probabilities (ERPs) of the various tests are presented in Figure 1.

One can see that MR successfully controls the probability of a type 1 error for $\Delta_1 = 0$ but is otherwise liberal. This problem increases as the correlation ρ decreases.

By design, the other three tests are conservative at $\Delta_1 = 0$, with their rejection probability increasing with Δ_1.

CE can also be liberal for $\rho \in \{-0.5, 0.0\}$, though much less compared to MR.

Both TwoStep and Cons are successful in controlling the probability of a type 1 error across all values of Δ_1 and ρ, with Cons approaching the nominal level α slightly faster.

5.2 Null behavior for $N = 10$

We continue with the more relevant case of $N = 10$ and consider two types of dependence structure for the return differentials. Specifically, the return differentials are generated as

$$\boldsymbol{d}_t \sim N(\boldsymbol{\Delta}, \boldsymbol{\Omega}) \ ,$$

where $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_{10})'$ and $\boldsymbol{\Omega}$ is either the identity covariance matrix of dimension 10 or a Toeplitz covariance matrix of dimension 10 with typical element $\omega_{i,j} = 0.9^{|i-j|}$. In the former case, the return differentials of a common period t are mutually independent; in the latter case, there is strong dependence for return differentials close to each other and weak dependence for return differentials far apart from each other.

We consider three basic designs for $\boldsymbol{\Delta}$:

- **(D1)** $\boldsymbol{\Delta} = (\Delta, \ldots, \Delta, -\Delta/10, \ldots, -\Delta/10)'$
- **(D2)** $\boldsymbol{\Delta} = (\Delta, \ldots, \Delta, -\Delta)'$
- **(D3)** $\boldsymbol{\Delta} = (\Delta, \ldots, \Delta, 0)'$

with $\Delta \in \{0, 0.025, 0.05, \ldots, 0.45, 0.475, 0.5\}$.

When $\Delta = 0$, then $\boldsymbol{\Delta}$ lies on the boundary of R_1 R_2 for all three designs. When $\Delta > 0$, then: for design 1 and design 2, $\boldsymbol{\Delta}$ lies in the interior of R_2; for design 3, $\boldsymbol{\Delta}$ lies on the boundary part of R_2 . But it never lies in R_3; that is, the alternative of strict monotonicity is never true.

For an illustration in terms of the corresponding vectors of expected returns, which have dimension $N + 1 = 11$, see Figure 2; the value $\Delta = 0.1$ is used there. The point of these particular designs is to get a feel of how anti-conservative the test of PT can turn in finite samples when the relation is not strictly monotonic but 'close' to strictly monotonic.

The empirical rejections probabilities (ERPs) of the various tests are presented in Figure 3 and Figure 4.

One can see that MR is often liberal for all three designs. In particular, for design 3 it is always liberal except for $\Delta = 0$. And for large values of Δ, the probability of a type 1 error rises as high as about 70% in the independent case. But even in design 1, where half of the expected return differentials are actually negative, albeit at a smaller magnitude, the test is generally liberal and the probability of a type 1 error can be as high as 15%. The overrejection problem is less pronounced for the dependent case (that is, for the Toeplitz covariance matrix) but still noticeable. The fact that MR can be anti-conservative in finite samples even for parameters $\boldsymbol{\Delta}$ in the interior of R_2 is in accordance with Remark 4.5 and does not contradict Theorem 4.1.

CE is somewhat liberal in design 3 for the independent case, though not for the dependent case and also never in design 1 or design 2.

The performances of TwoStep and Cons are virtually indistinguishable. In design 3, the probability of a type 1 error tends to 0.05 as Δ increases; it never lies above 0.05. In design 1 and design 2, the probability of a type 1 error is rather flat near zero. This holds true both for the independent and the dependent case. Note here that since $\boldsymbol{\Delta}$ lies in the interior of R_2 in design 1 and design 2, rather than on the boundary part, the probability of a type 1 error is necessarily bounded away from 0.05 for CE, TwoStep, and Cons.

5.3 Alternative behavior for $N = 10$ and $N = 5$

We now turn to performance with respect to power instead.

Ceteris paribus, it is reasonable to expect that power increases when the number of return differentials, N, is decreased. If the applied researcher uses a very large number N, then it will be rather (more) difficult for her to establish a positive expected return differential for each category.

To examine this effect, we consider the case where the expected return differentials are all the same and employ $N = 10$ or $N = 5$. To keep everything else equal, the expected return differentials must be twice as large for $N = 5$ compared to $N = 10$. This results in what we call designs 4 and 5:

- **(D4)** $\boldsymbol{\Delta} = (\Delta, \ldots, \Delta)'$ with $\Delta \in \{0, 0.025, 0.05, \ldots, 0.45, 0.475, 0.5\}$.
- **(D5)** $\boldsymbol{\Delta} = (\Delta, \Delta, \Delta, \Delta, \Delta)'$ with $\Delta \in \{0, 0.05, 0.1, \ldots, 0.95, 1.0\}$.

For example, $\Delta = 0.2$ in design 4 with $N = 10$ corresponds to $\Delta = 0.4$ in design 5 with $N = 5$. In other words, by reducing the number of categories by half, the expected return differentials must get doubled.

Also for $N = 5$ we consider two cases for the covariance matrix $\mathbf{\Omega}$ of the return differentials: the identity covariance matrix of dimension 5 or a Toeplitz covariance matrix of dimension 5 with typical element $\omega_{i,j} = 0.8^{|i-j|}$.

In both designs 4 and 5, the parameter $\mathbf{\Delta}$ lies on the boundary of R_1 and R_2 when $\Delta = 0$ and in R_3 when $\Delta > 0$. So the alternative of strict monotonicity is generally true, with the exception of $\Delta = 0$.

The empirical rejections probabilities (ERPs) of the various tests are presented in Figure 5 and Figure 6. Naturally, MR has the highest power, followed by CE, followed by Cons and TwoStep.

One can see that for all tests the power increases when N is reduced from $N = 10$ to $N = 5$, but on relative terms the most for TwoStep and Cons. For the independent case, take the combination $(N = 10, \Delta = 0.2)$ in design 4 that corresponds, one-to-one, to the combination $(N = 5, \Delta = 0.4)$ in design 5. While before the power of those two tests in design 3 was slightly below 0.05, it has now risen to almost one in design 4.

It is also noteworthy to point out that the differences in power are generally much reduced for the case of dependent returns (that is, when the covariance matrix of the return differentials is a Toeplitz matrix); see Figure 6 versus Figure 5.

Applied researchers often construct either $N + 1 = 10$ or $N + 1 = 5$ categories based on portfolio sorts using either deciles or quintiles. For establishing strict monotonicity in expected asset returns, it is then preferable to use quintiles instead of deciles in terms of the power of the tests. On the other hand, if strict monotonicity can be established for a larger value of N, this can be considered more convincing evidence. If the choice of N is not dictated by the application it hand, then it is up to the applied researcher to use her judgment in selecting a suitable number.

Remark 5.1 We certainly do not aim to address the difficult problem on how to 'optimally' choose the number of categories, N. The point of the above discussion is merely to make clear that, ceteris paribus, power can be increased by decreasing N. On the other hand, the larger is N, the more 'convincing' is the established strict monotonicity, assuming the test still rejects.

Regardless, N must never be chosen in a data-dependent fashion, using the same data from which the test is computed. Such a choice would invalidate all tests discussed in this paper. ■

5.4 Overall recommendations

When it is not certain that the relation must be either weakly increasing or strictly increasing, the MR test should not be used. The test can decide in favor of a strictly increasing relation

with probability (much) exceeding the nominal significance level when the true relation is non-monotonic or weakly increasing.

The CE test has related problems, though to a much smaller extent. Still, we do not recommend it for general use.

The only two tests that are safe to use in general are the Two-Step and the Cons tests. The performance of these two tests is rather similar. Since the Cons test is much easier to implement, it is the one we recommend for practical use.

The situation is reversed when the goal is only to establish the *direction* of monotonicity as strictly increasing (versus weakly decreasing). In this case, the MR test successfully controls the probability of a type 1 error. And since the MR test has the highest power of all four tests, it is then the preferred one.

Therefore, applied researchers should have both the MR test and the Cons test in their financial econometrics toolbox.

6 Empirical applications

In this section, we revisit two of the empirical applications of PT before considering some additional new ones.

6.1 Revisiting two empirical applications of PT

PT present two main empirical applications: one application concerning equity returns based on portfolios formed on CAPM beta and another application concerning bond term premia based on maturity. Their two corresponding plots of sample average returns and sample average term premia, respectively, according to return categories are reproduced in Figure 7.

We first turn to the application concerning equity returns. As detailed in their Subsection 4.1, the MR test of PT decides in favor of a strictly increasing relation, that is, in favor of the alternative $\boldsymbol{\Delta} \in R_3$. In contrast, since it is clear from the upper half of Figure 7 that $\hat{\Delta}_T \in R_2$, none of the new tests presented in this paper decide in favor of $\boldsymbol{\Delta} \in R_3$; see Remark 3.1. Instead, all these tests 'accept' the null hypothesis of $\boldsymbol{\Delta} \in R_1 \cup R_2$. In view of these differing results, the question then becomes whether one takes for granted that the relation between the CAPM beta of a stock and its expected return must be monotonic (as implied by the CAPM, for example); see Remark 2.3.

We next turn to the application concerning bond term premia. As detailed in their Subsection 4.3, the MR test of PT does not decide in favor of a strictly increasing relation, that is, in favor of the alternative $\boldsymbol{\Delta} \in R_3$. Since it is clear from the lower half of Figure 7 that $\hat{\Delta}_T \in R_2$, the new tests presented in this paper all agree with this conclusion.

6.2 Additional empirical applications

We present three additional empirical applications where we consider the effect of short-term reversal; here, short-term reversal is one-month reversal. We do this separately for small-size firms, mid-size firms, and large-size firms. To this end, we use a data set provided by Kenneth R. French labeled "25 Portfolios Formed on Size and Short-Term Reversal (5 x 5)".[2] In particular, we consider monthly average value-weighted returns from 01/1927 until 12/2010. In total, there are 25 portfolios. We use the first five, which correspond to quintile portfolios formed on short-term reversal for small-size firms, the middle five, which correspond to quintile portfolios formed on short-term reversal for mid-size firms, and the last five, which correspond to quintile portfolios formed on short-term reversal for large-size firms. The sample size is $T = 1,008$.

The test statistics are obtained as

$$t_{T,i} \equiv \frac{\hat{\Delta}_{T,i}}{\hat{\sigma}_{T,i}} \quad \text{where} \quad \hat{\Delta}_{T,i} \equiv \bar{d}_{T,i} \equiv \frac{1}{T}\sum_{t=1}^{T} d_{t,i} \tag{6.1}$$

and the standard errors $\hat{\sigma}_{T,i}$ are computed via a kernel method using the quadratic-spectral kernel and the automatic choice of bandwidth proposed by Andrews (1991). The critical values of the tests are computed via Algorithm 4.1. We employ the circular block bootstrap of Politis and Romano (1992) with a block size of $b = 12$, based 10,000 bootstrap resamples always. The choice $b = 12$ seems justified as none of the data display strong serial correlation. (The qualitative results, in terms of rejecting the null hypothesis or not, remain unchanged if a block size of $b = 24$ is used instead.)

The average returns on quintile portfolios formed on short-term reversal are displayed in Figure 8. For all three firm sizes, the observed relation is strictly decreasing. However, the 'steepness' of the relation diminishes in magnitude as the firm size goes up.

Since the observed relations are strictly decreasing rather than increasing, all return differentials were multiplied by negative one before feeding the data to the testing procedures. The results of the MR and Cons tests, for a nominal level of $\alpha = 0.05$, are presented in Table 1. Both tests decide in favor of a strictly decreasing relation for small-size firms and fail to do so for large size-size firms. However, the test results differ for mid-size firms: the MR test decides in favor of a strictly decreasing relation while the Cons test does not.

7 Conclusions

In many instances, empirical research in finance seeks to address whether a strictly increasing relation exists between an asset's expected return and some underlying characteristic of interest. For example, in the context of equity returns, such a characteristic might be CAPM beta, size, book-to-market, momentum, or reversal; in the context of bond returns, such a characteristic might be maturity or rating quality. If this characteristic is ordered into more than

[2] Data available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

two categories, then a convincing test needs to establish strict monotonicity over all categories: a simple test of "high minus low", only comparing the two most extreme categories, is not convincing.

In a recent paper, Patton and Timmermann (2010) propose a new test, taking all categories of the underlying characteristic into account. Compared to previous related proposals, they are, to our knowledge, the first ones to postulate a strictly increasing relation as the alternative hypothesis of the test, rather than as the null hypothesis. This is the correct formulation if the goal is to establish strict monotonicity with a quantifiable statistical significance. In addition, they postulate a weakly decreasing relation as the null hypothesis. Compared to allowing, more generally, also for a non-monotonic or weakly increasing relation under the null, this approach results in a smaller critical value and thereby in higher power of the test. But if a non-monotonic or weakly increasing relation is not ruled out, the test can break down and falsely 'establish' a strictly increasing relation with high probability. Therefore, the test of Patton and Timmermann (2010) should only be used as a test for the *direction* of (existing) monotonicity.

In this paper, we have proposed some alternative tests that also allow for a non-monotonic or weakly increasing relation under the null and that successfully control the probability of a type 1 error. These tests do not require modeling the dependence structure of the time series nor the covariance matrix of the observed return differentials. A suitable bootstrap method is used to implicitly account for these unknown features of the underlying data generating mechanism. As is unavoidable, such tests have lower power compared to the test of Patton and Timmermann (2010). This is the price one has to pay to safe-guard against the possibility of falsely 'establishing' strict monotonicity (beyond the nominal significance level of the test) in general settings.

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A Connection to the quasi-likelihood ratio test

Suppose $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Omega})$, with $\boldsymbol{\Omega}$ a known positive definite matrix. For testing problem (2.10) applied to the vector $\boldsymbol{\mu}$, namely

$$H_0 : \boldsymbol{\mu} \in R_1 \cup R_2 \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \in R_3 \ , \tag{A.1}$$

the generalized quasi-likelihood ratio (QLR) test rejects for large values of

$$\frac{\sup_{\boldsymbol{\mu} \in \mathbb{R}^N} \exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right]}{\sup_{\boldsymbol{\mu} \in R_3^{\complement}} \exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right]} \ , \tag{A.2}$$

where $R_3^{\complement} = R_1 \cup R_2$. Assume that $\boldsymbol{X} > \boldsymbol{0}$, or the test accepts H_0 anyway. Since the supremum in the numerator occurs when $\boldsymbol{\mu} = \boldsymbol{X}$, the QLR test rejects for large values of

$$\inf_{\boldsymbol{\mu} \in R_3^{\complement}} (\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}(\boldsymbol{X}-\boldsymbol{\mu}) \ . \tag{A.3}$$

In the case where $\boldsymbol{\Omega}$ is a diagonal matrix, the infimum occurs at some $\boldsymbol{\mu}$ that satisfies

$$\mu_i = 0 \text{ for some } i \in \{1, \ldots, N\} \quad \text{and} \quad \mu_j = X_j \text{ for all } j \in \{1, \ldots, N\} \text{ with } j \neq i \ . \tag{A.4}$$

The resulting value of (A.3) is then $\min_i X_i^2/\omega_{i,i}$, and rejecting for large values is equivalent to rejecting for large $\min_i X_i/\sqrt{\omega_{i,i}}$.

In case $\boldsymbol{\Omega}$ is not known, but a suitable estimator $\hat{\boldsymbol{\Omega}} = \text{diag}(\hat{\omega}_{1,1}, \ldots, \hat{\omega}_{N,N})$ is available, a feasible version of the QLR test rejects for large values of $\min_i X_i/\sqrt{\hat{\omega}_{i,i}}$.

In the setup of the paper, Assumptions (A1)–(A3) imply the following approximate sampling distribution, at least for large sample sizes:

$$\hat{\boldsymbol{\Delta}}_T \mathrel{\dot{\sim}} N(\boldsymbol{\Delta}, \boldsymbol{\Omega}/T) \quad \text{or, equivalently,} \quad \sqrt{T}\hat{\boldsymbol{\Delta}}_T \mathrel{\dot{\sim}} N(\sqrt{T}\boldsymbol{\Delta}, \boldsymbol{\Omega}) \ , \tag{A.5}$$

where the symbol $\dot{\sim}$ denotes "is approximately distributed as".

Hence — with $\sqrt{T}\hat{\boldsymbol{\Delta}}_T$ playing the role of $\boldsymbol{X}$, with $\sqrt{T}\boldsymbol{\Delta}$ playing the role of $\boldsymbol{\mu}$, and with $\sqrt{T}\hat{\sigma}_{T,i}$ playing the role of $\hat{\omega}_{i,i}$ — the test statistic t_T^{min} of (3.4) can be considered a feasible QLR test statistic under the assumption of a diagonal covariance matrix $\boldsymbol{\Omega}$. And hence the Cons test of Subsection 3.1 can be interpreted as a feasible QLR test for testing problem (2.10).

The assumption of $\boldsymbol{\Omega}$ being diagonal does not generally hold true for our application. It is, therefore, natural to consider the following feasible QLR test statistic under an arbitrary covariance matrix $\boldsymbol{\Omega}$:

$$\inf_{\boldsymbol{\Delta} \in R_3^{\complement}} (\hat{\boldsymbol{\Delta}}_T - \boldsymbol{\Delta})'\hat{\boldsymbol{\Omega}}^{-1}(\hat{\boldsymbol{\Delta}}_T - \boldsymbol{\Delta}) \ , \tag{A.6}$$

where $\hat{\boldsymbol{\Omega}}$ is a suitable estimator of $\boldsymbol{\Omega}$.

Unfortunately, the choice of test statistic (A.6) results in a test which can be liberal in finite samples; at least when $\hat{\boldsymbol{\Omega}}$ is the natural estimator of $\boldsymbol{\Omega}$, namely the sample covariance matrix

of the observations $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_T$.[3] This finding is in line with previous results in the literature that show that tests based on quadratic test statistics using the inverse of a sample covariance matrix are often liberal in finite samples; for example, see Hayashi (2000, Section 3.5), Ren and Shimotsu (2009), and the references therein.

B Test based on minimum is UMP among monotone tests

As in Appendix A, assume $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Omega})$, where $\boldsymbol{\Omega}$ is a known positive definite matrix. Consider the testing problem (2.10) applied to the vector $\boldsymbol{\mu}$, namely

$$H_0 : \boldsymbol{\mu} \in R_1 \cup R_2 \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \in R_3 \ . \tag{B.1}$$

Restrict attention to monotone tests as described in Remark 3.1. Thus, if a test based on $\boldsymbol{X}$ rejects H_0, then it must also reject H_0 based on any $\boldsymbol{X}' \geq \boldsymbol{X}$.

Theorem B.1 *Consider the above testing problem. Among monotone level α (nonrandomized) tests with monotone increasing rejection region,*[4] *the test that rejects when $\min_i X_i/\sqrt{\omega_{i,i}} \geq z_{1-\alpha}$ is uniformly most powerful (UMP), where $z_{1-\alpha}$ is the $1-\alpha$ quantile of the standard normal distribution.*

Proof: Let $\mathcal{R} \subset \mathbb{R}^N$ be the rejection region of the claimed UMP test, so

$$\mathcal{R} \equiv \{\boldsymbol{x} : \min_i x_i/\sqrt{\omega_{i,i}} \geq z_{1-\alpha}\} \ . \tag{B.2}$$

Assume there exists another test with monotone increasing rejection region $\mathcal{R}'$ that has better power against even one alternative $\boldsymbol{\mu} \in R_3$. Then, it cannot be the case that $\mathcal{R}' \subseteq \mathcal{R}$, or the test based on $\mathcal{R}'$ could not have better power (as it would never reject unless the test based on $\mathcal{R}$ did). Hence, $\mathcal{R}'$ must include a point $\boldsymbol{y} \equiv (y_1, \ldots, y_N)' \notin \mathcal{R}$. Therefore, for at least some j, it holds true that $y_j < \sqrt{\omega_{j,j}} z_{1-\alpha}$. Let

$$\mathcal{R}'' \equiv \{\boldsymbol{x} : \boldsymbol{x} \geq \boldsymbol{y}\} \ . \tag{B.3}$$

Since $\mathcal{R}'$ is assumed monotone, it must be the case that $\mathcal{R}'' \subseteq \mathcal{R}'$. But then, the supremum of the probability of a type 1 error of the test based on $\mathcal{R}'$ satisfies

$$\sup_{\boldsymbol{\mu} \in R_1 \cup R_2} \mathbb{P}_{\boldsymbol{\mu}}\{\mathcal{R}'\} \geq \sup_{\boldsymbol{\mu} \in R_1 \cup R_2} \mathbb{P}_{\boldsymbol{\mu}}\{\mathcal{R}''\} \ . \tag{B.4}$$

The right hand side is bounded below by $\mathbb{P}_{\boldsymbol{\mu}}\{\mathcal{R}''\}$ with $\boldsymbol{\mu}$ satisfying $\mu_i = 0$ for some i and $\mu_j = B$ for all $j \neq i$, where B is some large value. As $B \to \infty$, this probability becomes

$$\mathbb{P}_{\mu_i=0}\{X_i \geq y_i\} > \mathbb{P}_{\mu_i=0}\{X_i \geq \sqrt{\omega_{i,i}} \cdot z_{1-\alpha}\} = \alpha \ . \tag{B.5}$$

[3] Results of corresponding Monte Carlo studies are not included in the paper, but are available from the authors upon request.

[4] A region $\mathcal{R} \subseteq \mathbb{R}^{\mathcal{N}}$ is called *monotone* (increasing) provided the following condition holds: if $\boldsymbol{x} \in \mathcal{R}$ and $\boldsymbol{x}' \geq \boldsymbol{x}$, then necessarily $\boldsymbol{x}' \in \mathcal{R}$ as well.

This is a contradiction, as the test would not be level α.

The argument actually applies to any family of distributions $F_{\boldsymbol{\theta}}$ that are stochastically increasing in a parameter $\boldsymbol{\theta}$, meaning $\mathbb{P}_{\boldsymbol{\theta}_1}\{\mathcal{R}\} \leq \mathbb{P}_{\boldsymbol{\theta}_2}\{\mathcal{R}\}$ whenever $\mathcal{R}$ is a monotone (increasing) region region and $\boldsymbol{\theta}_1 \leq \boldsymbol{\theta}_2$. This holds true for any multivariate location model.

In particular, under our maintained assumptions, an approximate multivariate normal location model applies to the vector of test statistics $\boldsymbol{t}_T = (t_{T,1}, \ldots, t_{T,N})'$, at least when the sample size T is large. And, therefore, in an approximate setup, the Cons test is UMP among all monotone level α tests for testing problem (2.10).

Remark B.1 Consider the multivariate normal model of this section, specializing to the case where $\boldsymbol{\Omega}$ is the identity matrix. Testing problem (2.8) applied to the vector $\boldsymbol{\mu}$ becomes

$$H_0 : \boldsymbol{\mu} \in R_1 \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \in R_3 \ . \tag{B.6}$$

The exact finite sample analogue of the MR test of PT in this case would be to reject when $\min X_i \geq c_N(1-\alpha)$, where the critical value is determined by

$$\mathbb{P}_{\boldsymbol{\mu}=\mathbf{0}}\{\min_i X_i \geq c_N(1-\alpha)\} = \alpha \ . \tag{B.7}$$

Therefore, $[1 - \Phi(c_n(1-\alpha))]^N = \alpha$, or

$$c_N(1-\alpha) = \Phi^{-1}\big(1 - \alpha^{1/N}\big) \ . \tag{B.8}$$

It follows that, if the correct null hypothesis is instead specified by (B.1), the probability of a type 1 error is maximized by maximizing

$$\mathbb{P}_{\boldsymbol{\mu}}\{\min X_i \geq c_N(1-\alpha)\} \tag{B.9}$$

over $\boldsymbol{\mu} \in R_1 \cup R_2$. This probability is maximized when one $\mu_i = 0$ and the remaining μ_j tend to ∞. It follows that the maximum probability of a type 1 error of the MR test is exactly

$$\mathbb{P}_{\mu_1=0}\Big\{X_1 \geq \Phi^{-1}\big(1-\alpha^{1/N}\big)\Big\} = 1 - \Phi\Big[\Phi^{-1}\big(1-\alpha^{1/N}\big)\Big] = \alpha^{1/N} \ , \tag{B.10}$$

which is bigger than α, unless $N = 1$, and it approaches one as N increases. Clearly, such a test does not control the type 1 error for our formulation (B.1) of the testing problem and is much too liberal. ■

C Joint confidence region for $\boldsymbol{\Delta}$

This section briefly summarizes the single-step method described in Romano and Wolf (2005) to compute the upper limits $\hat{\boldsymbol{u}}_T \equiv (\hat{u}_{T,1}, \hat{u}_{T,2}, \ldots, \hat{u}_{T,N})'$ utilized in Subsection 3.3. Specifically, they are of the form

$$\hat{u}_{T,i} \equiv \hat{\Delta}_{T,i} + \hat{d}_1 \cdot \hat{\sigma}_{T,i} \ . \tag{C.1}$$

Here, $\hat{d}_1$ is a (consistent) estimate of d_1, which is defined as the $1-\beta$ quantile of the sampling distribution of the random variable

$$\max_i \frac{(\hat{\Delta}_{T,i} - \Delta_i)}{\hat{\sigma}_{T,i}} \ . \tag{C.2}$$

The following algorithm details how d_1 is approximated via the bootstrap.

Algorithm C.1 (Computation of $\hat{d}_1$)

- Generate bootstrap data $\{\boldsymbol{r}_1^*, \boldsymbol{r}_2^*, \ldots, \boldsymbol{r}_T^*\}$ and compute statistics $\hat{\Delta}_{T,i}^*$ and $\hat{\sigma}_{T,i}^*$ from these data, for $i = 1, \ldots, N$
- Compute $\max_T^* \equiv \max_i (\hat{\Delta}_{T,i}^* - \hat{\Delta}_{T,i})/\hat{\sigma}_{T,i}^*$
- Repeat this process B times, resulting in statistics $\max_{T,1}^*, \ldots, \max_{T,B}^*$
- The $1-\beta$ empirical quantile of these B statistics is the bootstrap estimate $\hat{d}_1$

For motivation and further details, the reader is referred to Romano and Wolf (2005, Section 4).

D Proofs

Proof of Theorem 4.1: Denote the correlation matrix corresponding to the limiting covariance matrix $\boldsymbol{\Omega}$ of Assumption (A1) by $\boldsymbol{\Pi}$, with typical element $\pi_{i,j}$, that is,

$$\pi_{i,j} = \frac{\omega_{i,j}}{\sqrt{\omega_{i,i} \cdot \omega_{j,j}}} \ . \tag{D.1}$$

Define centered statistics by

$$v_{T,i} \equiv \frac{\hat{\Delta}_{T,i} - \Delta_i}{\hat{\sigma}_{T,i}} \ , \tag{D.2}$$

and let $\boldsymbol{v}_T \equiv (v_{T,1}, \ldots, v_{T,N})'$. Then $\boldsymbol{v}_T$ converges in distribution to $N(\boldsymbol{0}, \boldsymbol{\Pi})$. Let $\boldsymbol{y} \equiv (y_1, \ldots, y_N)'$ be a random variable with distribution $N(\boldsymbol{0}, \boldsymbol{\Pi})$ and let $y^{min} \equiv \min_i y_i$. Also, denote by $c_{1-\alpha}$ the $1-\alpha$ quantile of the distribution of y^{min}.

Furthermore, since

$$t_{T,i} \equiv \frac{\hat{\Delta}_{T,i}}{\hat{\sigma}_{T,i}} = v_{T,i} + \frac{\Delta_{T,i}}{\hat{\sigma}_{T,i}} \ , \tag{D.3}$$

it holds that

$$t_{T,i} \text{ converges in distribution to } \begin{cases} -\infty & \text{if } \Delta_i < 0 \\ N(0,1) & \text{if } \Delta_i = 0 \\ \infty & \text{if } \Delta_i > 0 \end{cases} \ . \tag{D.4}$$

(Here, in the first case and in the third case, convergence in distribution is actually equivalent to convergence in probability.)

To prove part (ia), first consider the case $\boldsymbol{\Delta} = \boldsymbol{0}$. It then follows that t_T^{min} converges in distribution to y^{min} and that the critical value of the test converges to $c_{1-\alpha}$ in probability. Hence, the rejection probability converges to α. Any other parameter $\boldsymbol{\Delta}$ considered satisfies

$\min_i \Delta_i < 0$, in which case $t_T^{\min}$ converges to $-\infty$ in probability, resulting in a limiting rejection probability of zero.

To prove part (ib), any parameter $\boldsymbol{\Delta}$ considered satisfies $\min_i \Delta_i = 0$ and $\max_i \Delta_i > 0$. Let $I(\boldsymbol{\Delta}) \equiv \{i : \Delta_i = 0\}$. Then t_T^{min} converges in distribution to $y_{I(\boldsymbol{\Delta})}^{min} \equiv \min_{i \in I(\boldsymbol{\Delta})} y_i$. Note that $y_{I(\boldsymbol{\Delta})}^{min}$ is stochastically larger than y^{min}, since $I(\boldsymbol{\Delta})$ is a strict subset of $\{1, \ldots, N\}$. The rejection probability thus converges to $\mathbb{P}\{y_{I(\boldsymbol{\Delta})}^{min} > c_{1-\alpha}\} > \mathbb{P}\{y^{min} > c_{1-\alpha}\} = \alpha$.

To prove part (ic), any parameter $\boldsymbol{\Delta}$ considered satisfies $\min_i \Delta_i > 0$, in which case $t_T^{\min}$ converges to ∞ in probability, resulting in a limiting rejection probability of one.

To prove part (iia), any parameter $\boldsymbol{\Delta}$ considered satisfies $\min_i \Delta_i \leq 0$. To start out, consider the Cons test. The critical value converges to $z_{1-\alpha}$ in probability. But, in the limit, the distribution of the test statistic t_T^{min} is (weakly) stochastically smaller than the standard normal distribution for all values of $\boldsymbol{\Delta}$ considered. So the limiting rejection probability of the test cannot exceed $1 - \alpha$. Next, consider the CE test. Since $\hat{\boldsymbol{\Delta}}_T$ converges in to $\boldsymbol{\Delta}$ in probability, the following is not difficult to show. First, if $\min_i \Delta_i < 0$, then the limiting rejection probability is zero; see Remark 3.1. Second, if $\boldsymbol{\Delta}$ contains one zero entry and the remaining entries are positive, the critical value converges to $z_{1-\alpha}$ in probability and the limiting rejection probability is equal to α. Third, if $\boldsymbol{\Delta}$ contains several zero entries and the remaining (if any) entries are positive, let again $I(\boldsymbol{\Delta}) \equiv \{i : \Delta_i = 0\}$. The test statistic $t_T^{\min}$ converges in distribution to $\min_{i \in I(P)} y_i$. Denote the $1 - \alpha$ quantile of the distribution of $\min_{i \in I(P)} y_i$ by $c_{I(P),1-\alpha}$. Then, for any $\epsilon > 0$, the probability that the critical value of the test will be smaller than $c_{I(P),1-\alpha} - \epsilon$ converges to zero. As a result, the limiting rejection probability of the test cannot exceed α. Finally, consider the Two-Step test. Denote by E the set in the sample space on which the joint confidence region (3.8) contains the true parameter $\boldsymbol{\Delta}$. Hence, on the set E, the employed critical value is (weakly) larger compared to computing the critical value under the true parameter. As a result, the limiting rejection probability cannot exceed the nominal level of the test, which is $\alpha - \beta$. Assume without loss of generality that on the set $E^{\complement}$ (that is, on the complement of the set E), the test always rejects. By design of the joint confidence region (3.8), the limiting probability of $E^{\complement}$ cannot exceed β. In sum, the limiting rejection probability cannot exceed $\alpha - \beta + \beta = \alpha$.

To prove part (iib), any parameter $\boldsymbol{\Delta}$ considered satisfies $\min_i \Delta_i > 0$, in which case $t_T^{\min}$ converges to ∞ in probability, resulting in a limiting rejection probability of one. Note here that for all three tests considered, the critical value converges to $z_{1-\alpha}$ in probability. ■

Small Size, $t_T^{min} = 2.02$

Test	MR	Cons
Critical Value	0.20	1.60

Mid Size, $t_T^{min} = 1.33$

Test	MR	Cons
Critical Value	0.13	1.67

Large Size, $t_T^{min} = 0.10$

Test	MR	Cons
Critical Value	0.17	1.88

Table 1: Test results for equity returns corresponding to the three panels of Figure 8. The nominal level is $\alpha = 0.05$. All return differentials were multiplied by negative one before feeding the data to the two testing procedures.

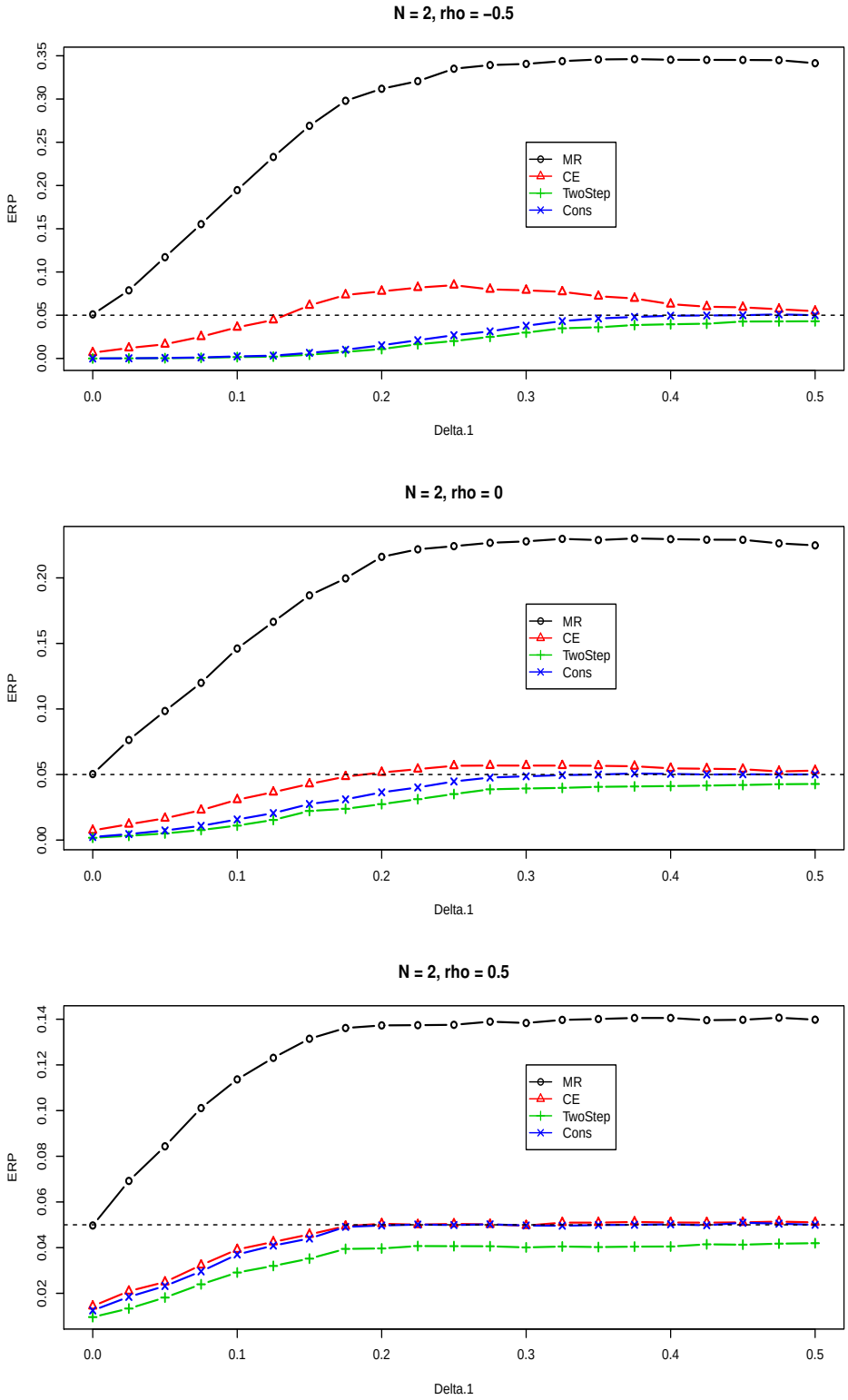


Figure 1: Empirical rejection probabilities (ERPs) for various tests in dimension $N = 2$ as a function of Δ_1, where $\boldsymbol{\Delta} = (\Delta_1, 0)$.

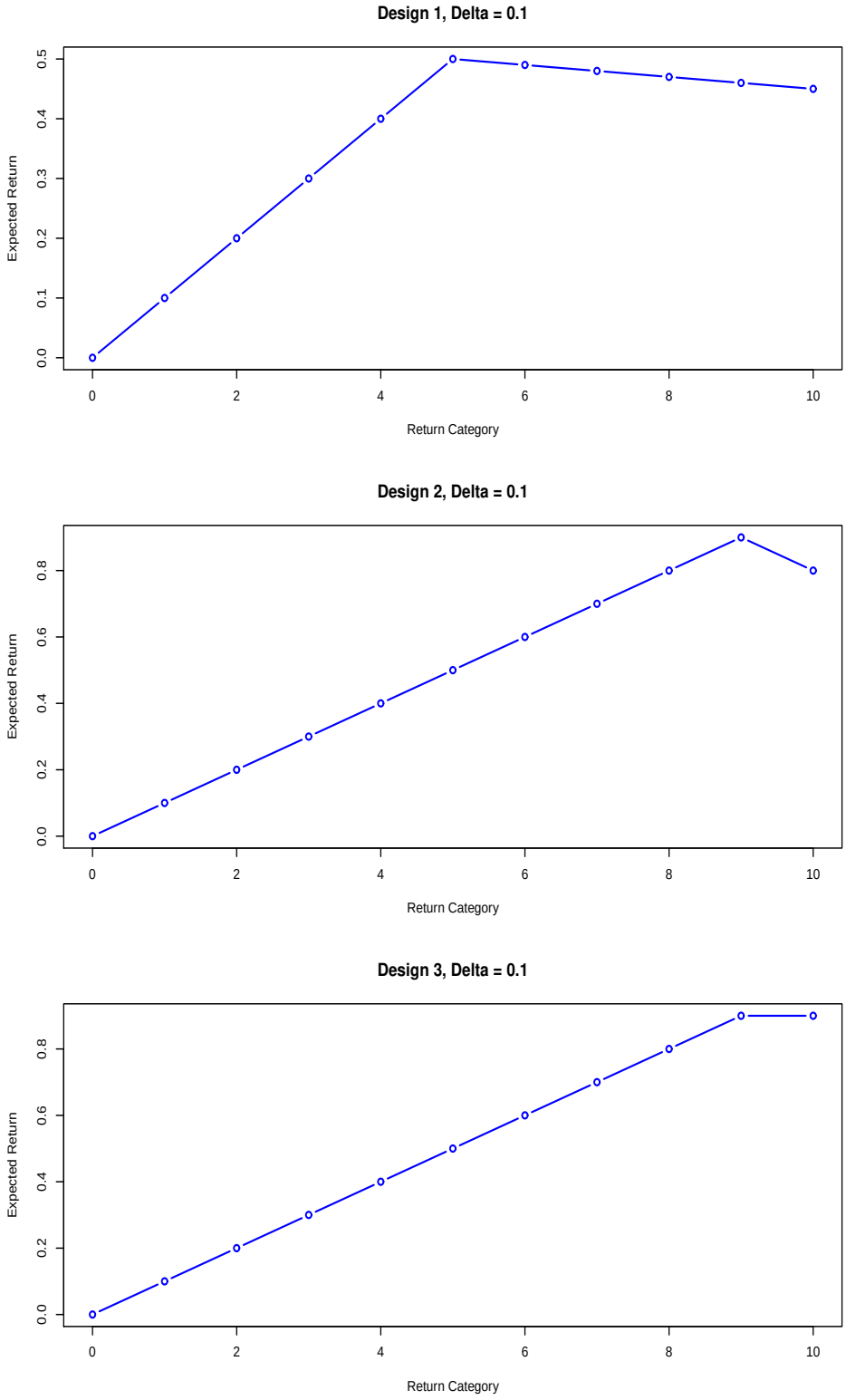


Figure 2: Vectors of expected returns, for $\Delta = 0.1$, corresponding to the three null designs detailed in Subsection 5.2.

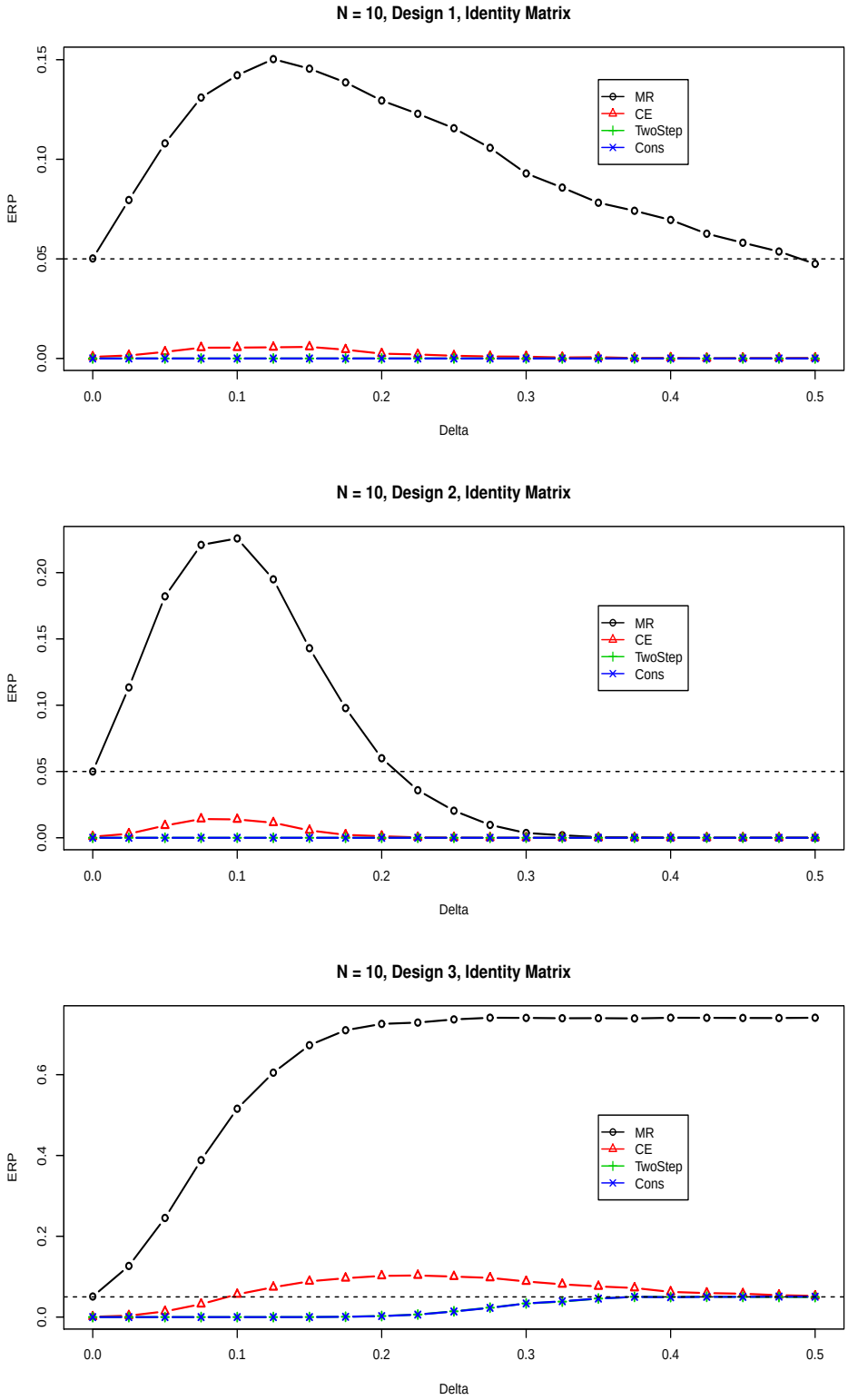


Figure 3: Empirical rejection probabilities (ERPs) under H_0 for various tests in dimension $N = 10$ as a function of Δ, where the three designs are detailed in Subsection 5.2. The covariance matrix of the return differentials is the identity matrix.

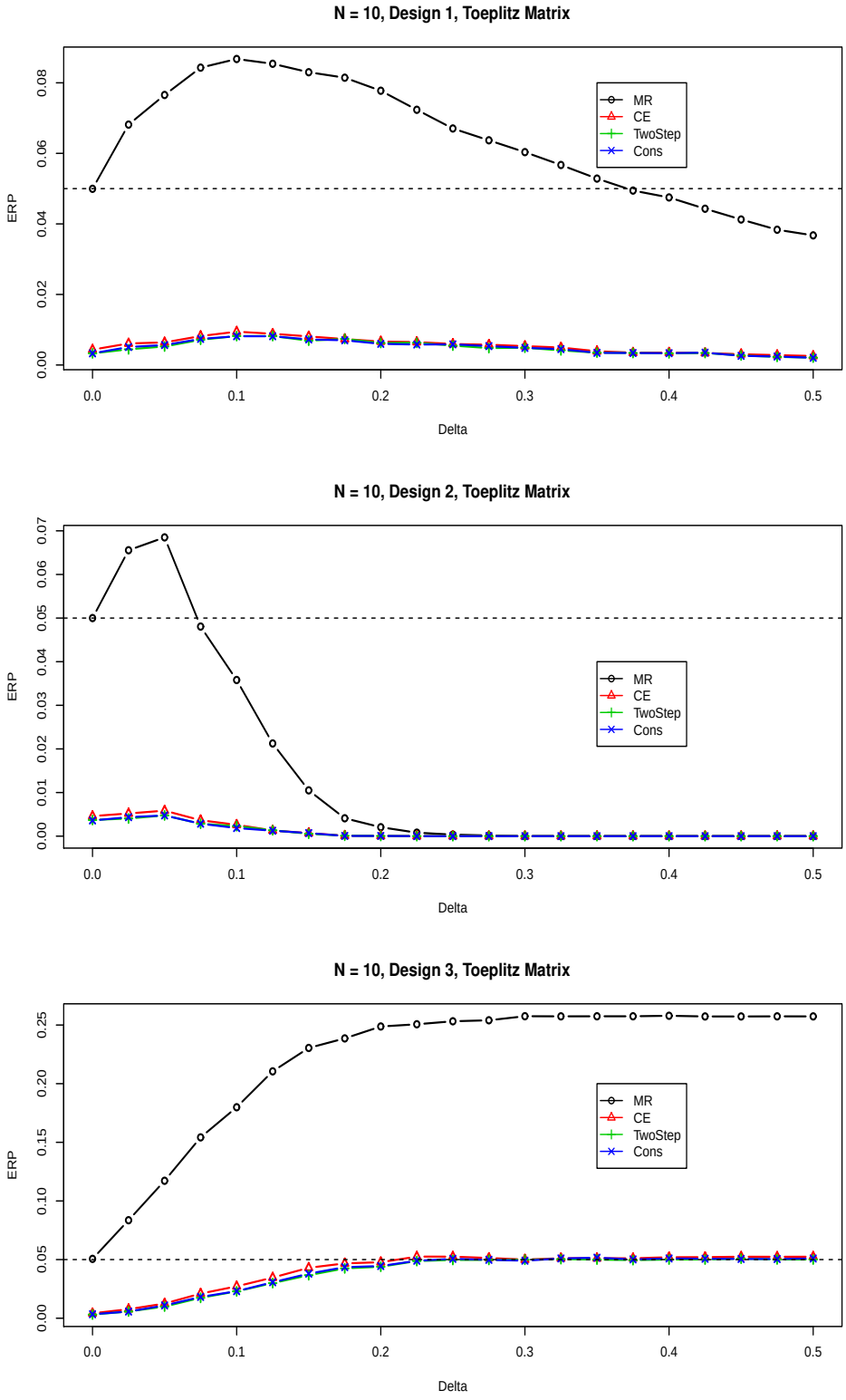
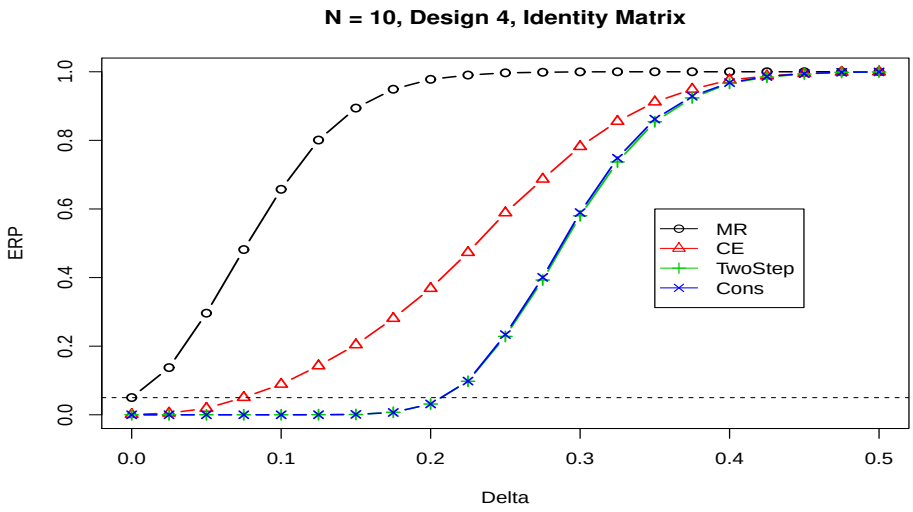


Figure 4: Empirical rejection probabilities (ERPs) under H_0 for various tests in dimension $N = 10$ as a function of Δ, where the three designs are detailed in Subsection 5.2. The covariance matrix of the return differentials is a Toeplitz matrix.



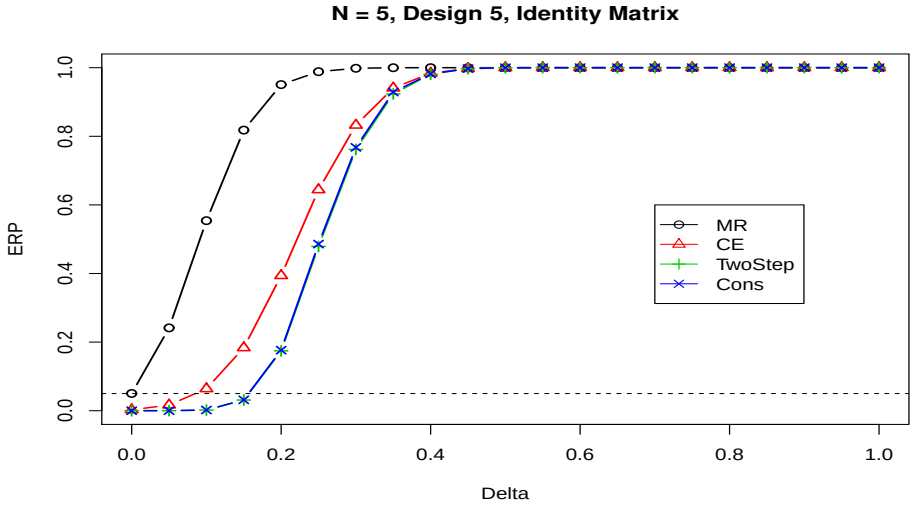


Figure 5: Empirical rejection probabilities (ERPs) under H_1 for various tests in dimensions $N = 10$ and $N = 5$ as a function of Δ, where the two designs are detailed in Subsection 5.3. The covariance matrix of the return differentials is the identity matrix.

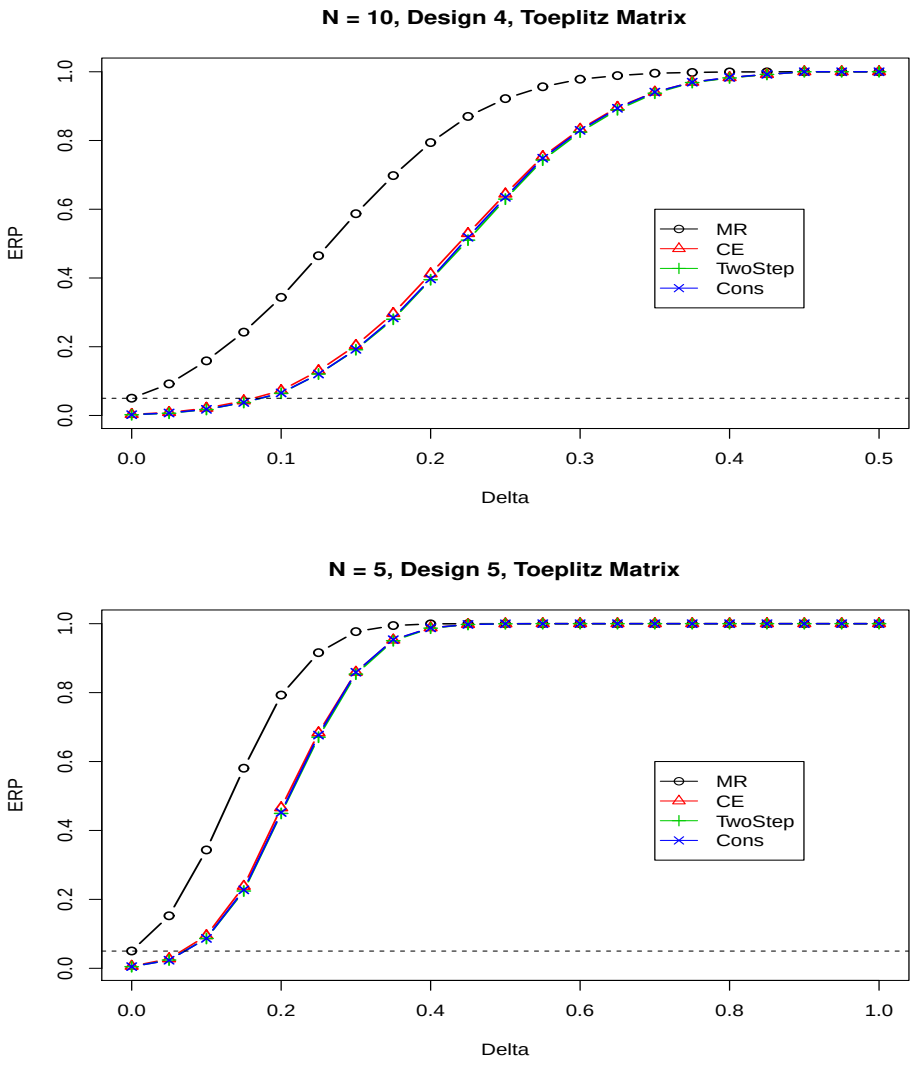


Figure 6: Empirical rejection probabilities (ERPs) under H_1 for various tests in dimensions $N = 10$ and $N = 5$ as a function of Δ, where the two designs are detailed in Subsection 5.3. The covariance matrix of the return differentials is a Toeplitz matrix.

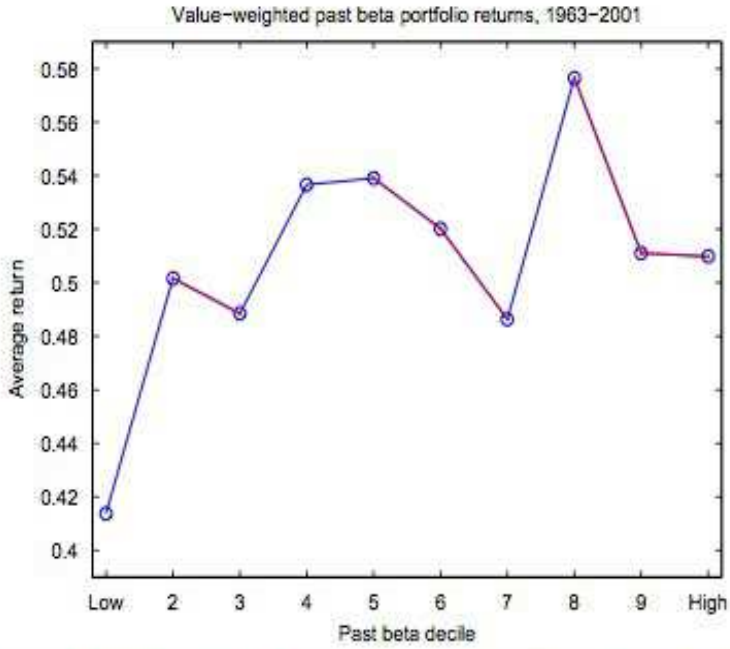


Fig. 1. Average monthly returns on decile portfolios formed on past 12-month Capital Asset Pricing Model (CAPM) beta, from July 1963 to December 2001.

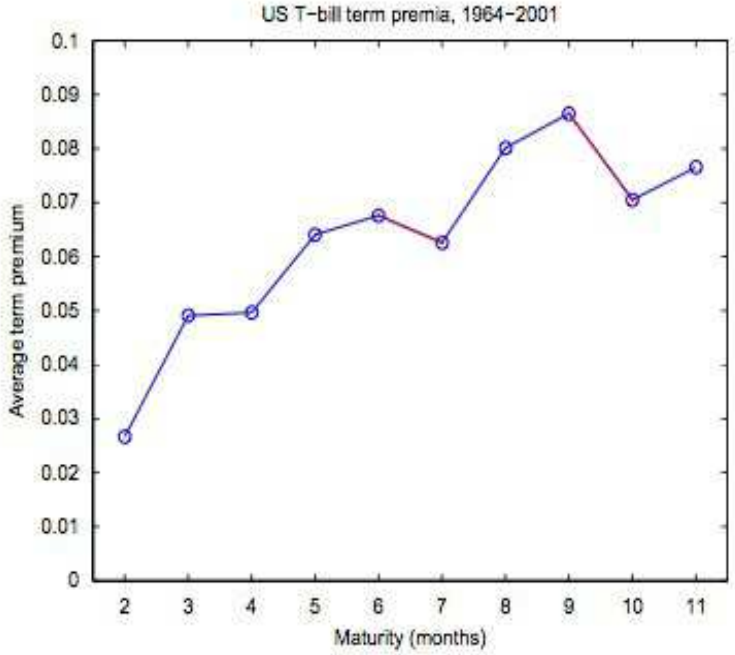


Fig. 2. Average monthly term premia for US T-bills, relative to a T-bill with one month to maturity, over the period January 1964 to December 2001.

Figure 7: Reproduction of Figure 1 and Figure 2 of Patton and Timmermann (2010).

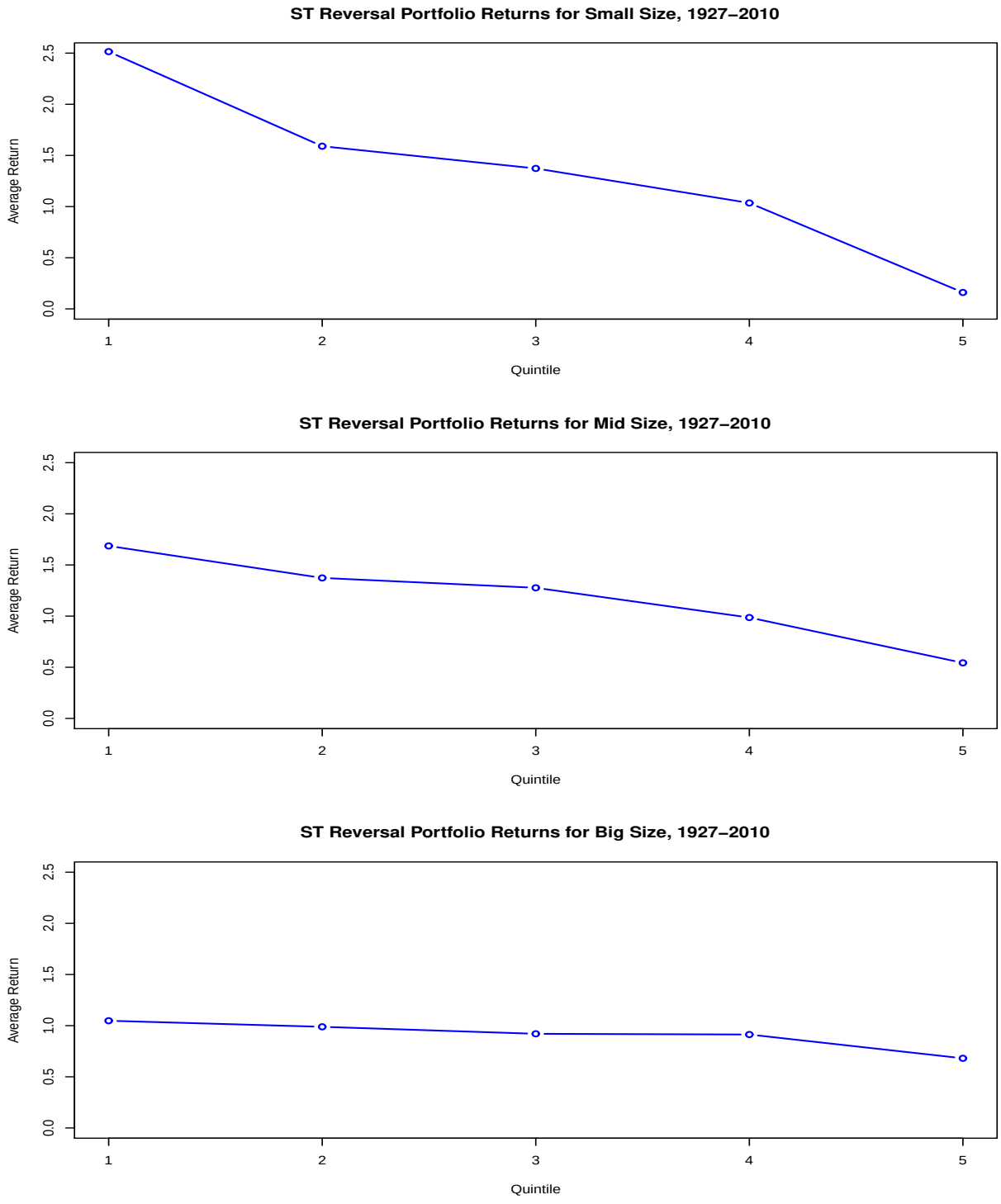


Figure 8: The upper panel displays average monthly value-weighted returns on quintile portfolios formed on short-term reversal for small-size firms, from January 1927 to December 2010. Similar for the middle and lower panel; except that mid-size firms and big-size firms, respectively, are used instead.